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EXTRAIT

Thermoelastic buckling, An unstable thermodynamic equilibrium at minimum entropy by M. A. BIOT



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by M. A. BIOT

Abstract. — The instability of thermodynamic equilibrium of a system in a state of minimum entropy is analyzed by treating the particular example of thermoelastic buckling of a thin elastic plate in an initial state of compression in its own plane. There are two critical buckling loads, one for slow isothermal deformation, the other for fast adiabatic deformation. For an intermediate load the buckling occurs at a finite rate controlled by the thermal conductivity of the elastic material. Thus a purely thermoelastic creep buckling is obtained. The general analysis is based on Lagrangian thermodynamics in the vicinity of an equilibrium state with initial stress. The general theory is compared with the solution derived from classical differential equations of thermoelasticity. The instability is non-oscillatory in accordance with a general theorem. Due to the phenomenological nature of the general theory, results are applicable to porous solids or materials which exhibit mechanical relaxation associated with internal diffusion, phase change, and mass transport.

1. INTRODUCTION

The phenomenon of elastic instability of a thin elastic rod is well known in the classical litterature of the theory of Elasticity. Euler's famous formula provides the critical value of the compression beyond which the elastic rod becomes unstable and exhibits a spontaneous sudden lateral deflection known as buckling. However there is a fundamental point which has been overlooked in the classical treatment. Euler's formula is expressed in term's of Young's modulus which measures the elastic rigidity of the material but does not take into account the fact that this elastic modulus exhibits two different values. One value which we call the isothermal modulus measures the rigidity for very slow elastic deformations where the temperature remains constant so that the elastic medium remains all the time in thermal equilibrium with the surroundings. The other value is the adiabatic modulus where the elastic deformation is very fast so that no heat is exchanged with the surroundings. For most materials an adiabatic extension is associated with a cooling and an adiabatic compression with a heating. Such materials are obviously more rigid for adiabatic deformations than for isothermal ones, hence the adiabatic modulus is larger than the isothermal modulus. Thus Euler's formula yields two values, an isothermal buckling load corresponding to the isothermal modulus and an adiabatic buckling load which is larger and corresponds to the adiabatic modulus. The question therefore arises which modulus to use in Euler's formula for the critical buckling load and what is the actual physical behavior of the system.

It should be immediately evident that the elastic rod will tend to buckle as soon as the isothermal critical load is reached. However for this value the buckling must be very slow to allow temperatures to remain constant throughout. If this were not the case the deformation would tend to become adiabatic with a corresponding rise of the critical load.

The buckling being a bending deformation the temperature is increased on the compressed side and decreased on the tension side. The buckling will continue only of these temperares are allowed to become equalized by thermal conduction in the material.

On the other hand if we apply the adiabatic critical load and neglect the inertia due to the mass of the material, the buckling is an instantaneous collapse.

Obviously for a compressive load, between the isothermal and the adiabatic, the buckling is a gradual purely thermoelastic *creep buckling* where the deflection rate is controlled by the thermal conductivity of the material. The problem was analyzed some time ago [1-2] by using the analogy between thermoelasticity and the theory of porous solids containing a viscous fluid. The pressure in the pores and the rate of flow of the fluid between the pores correspond to the temperature and heat flow of the thermoelastic material. The thermoelastic problem however is a particular case of the general irreversible thermodynamics of initially stressed continua, discussed extensively in a book [3].

The theory is based on the author's Lagrangian thermodynamics of linear irreversible processes [4-6] applied to a system in the vicinity of an instable equilibrium. Althought it is implicit in the earlier theory, a recent general analysis of non-linear thermoelasticity [7] points out more explicitly that the unstable equilibrium corresponds to a state of *minimum entropy*. Hence the instability fits into a model based on *probabilistic concepts of entropy and statistical mechanics*. The subject of non-linear thermoelasticity from an entirely different and classical viewpoint has been treated by a number of authors [8-10].

Our purpose here is to illustrate the fundamental physics involved by analyzing a simple problem represented by the buckling of a thin rectangular plate compressed in a direction parallel to its short edge. It is contained in a rigid thermally insulated box, large enough so that a fluid inside the box may be treated as a isothermal bath.

In section 2 it is shown that when the plate is maintained in its flat position of unstable equilibrium the whole isolated system is in a state of minimum entropy. The gradual buckling for a compressive load between the isothermal and adiabatic value is analyzed in section 3 on the basis of the author's Lagrangian thermodynamics. In section 4 it is shown that the result may also be derived on the basis of the classical differential field equation of thermoelasticity. The effect of inertia forces is introduced in section 5. A fundamental property of the type of unstable thermodynamic equilibrium considered here is the *non-oscillatory* nature of the instability. In other words the perturbation amplitudes are proportional to a real increasing exponential exp (pt) where p is real and positive. The proof which was established eatlier [3, page 442] is briefly outlined in section 6.

In section 7 attention is called to the generality of the present treatment. While it is restricted to the context of thermal diffusion it is applicable in its general form to problems including molecular diffusion.

2. THERMOELASTIC POTENTIAL AND MINIMUM ENTROPY

We consider a thin rectangular plate with edges parallel to the coordinate axes. The y axis is oriented across the thickness a and the planes $y = \pm a/2$ are the faces. The origin of x is taken at the center so that $x = \pm l/2$ represents the top and bottom edges, hence

l is the length along x between edges The width oriented along the third direction is arbitrary but large, and the deformation is assumed two-dimensional and represented by plane strain in the x, y plane. The plate is initially loaded by a uniform compressive stress P directed along x. The deformation considered is a bending of the plate due to buckling. The plate is assumed to be confined in a large thermally insulated rigid box, with the rigid walls producing an initial compressive stress P if the plate is maintained in its flat equilibrium position. The plate may rotate freely at its edges $x = \pm l/2$ through which the walls of the box exert the compressive stress P. We consider the thin plate theory of bending to be valid with the Kirchhoff-Bernouilli assumption that the plane initially normal to x remains normal to the line initially along x, i.e. to the neutral axis. The deflection of this line is denoted by

$$w = w(x) \tag{2.1}$$

The strain component along x is

$$\varepsilon_{11} = e_{xx} + \eta_{xx} \tag{2.2}$$

where

$$e_{xx} = -y \frac{d^2 w}{dx^2} \tag{2.3}$$

$$\eta_{xx} = \frac{1}{2l} \int_{-l/2}^{+l/2} \left(\frac{dw}{dx}\right)^2 dx$$

The term e_{xx} is due to the bending curvature while η_{xx} is due to the lengthening of the plate when it deflects between the fixed points $x = \pm l/2$.

The plate is surrounded by an isothermal bath at the temperature T_r , which is also inside the rigid box and its volume is assumed to be large enough so that T_r remains constant.

Since the deformation is two-dimensional in the $x \,.\, y$ plane we may consider a volume of the plate and of the corresponding isothermal bath of unit dimension in the direction normal to the $x \,.\, y$. plane. We shall refer to this volume as the plate and the isothermal bath, respectively.

We consider now the following non-classical thermodynamic potential

$$V = \int_{A} v dA = \int_{A} (u - T_r s) dA$$
 (2.4)

where u and s are the local internal energy and entropy per unit volume of the plate and A is the rectangular area of boundaries $x = \pm l/2$, $y = \pm a/2$. The definition includes the case of non uniform temperature of the system. The total internal energy and entropy of the plate, are

$$\mathbf{U} = \int_{\mathbf{A}} u d\mathbf{A} \quad \mathbf{S} = \int_{\mathbf{A}} s d\mathbf{A} \tag{2.5}$$

hence also

$$V = U - T_r S \tag{2.6}$$

The potential (2.4) for systems at non uniform temperature was introduced by the author in 1954 [4]. It was applied extensively [3, 5-6] and referred to as the *thermoelastic potential* [6-7, 11-12 page 166] in the particular case of thermoelasticity. These quantities are defined in such a way that u = s = 0 for the system in initial equilibrium at uniform temperature T_r and zero deformation w = 0.

An important property of the potential V is derived by applying the first principle, of energy conservation in the form

$$\mathbf{U} + \mathbf{H}_r = 0 \tag{2.7}$$

where H, is the heat energy acquired by the isothermal bath. Hence

$$S_r = \frac{H_r}{T_r} = -\frac{U}{T_r}$$
(2.8)

where S_r is the increase of entropy of the isothermal bath. It is assumed here that the kinetic energy is negligible either because the mass density of the plate is very small or because the motion is slow enough. The influence of the inertia force will be considered in section 5. Combining relations (2.6) and (2.8) we derive [4-6],

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where

$$V = -S' I_r$$
(2.9)
$$S' = S + S_r$$
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(a, a)

is the entropy of the total isolated system composed of the plate and isothermal bath. If

$$\mathbf{V} > \mathbf{0} \tag{2.10}$$

S' is negative which cannot happen according to the second principle, hence the system is stable. However if

$$\mathbf{V} < \mathbf{0} \tag{2.11}$$

the entropy S' may increase, and the equilibrium is unstable due to the fact that it represents a state of *minimum entropy*.

Consider now the value of v for an element of unit volume. We may write

$$dv = du - T_r ds \tag{2.12}$$

according to the first principle.

$$du = (-P + t_{11})d\varepsilon_{11} + dh$$
 (2.13)

where dh is the heat absorbed by the element and t_{11} is the incremental tensile stress added to the initial compression P. Also

$$(\mathbf{T}_r + \theta)ds = dh \tag{2.14}$$

where θ is the increase in temperature above T_r.

With the values (2.13) and (2.14) of du and dh expression (2.12) becomes.

$$dv = (-\mathbf{P} + t_{11})d\varepsilon_{11} + \theta ds \qquad (2.15)$$

With the value (2.2) of ε_{11} and neglecting third order terms we obtain

$$dv = -Pd\eta_{xx} + t_{11}de_{xx} + \theta ds \qquad (2.16)$$

Finally we put

$$dv' = t_{11}de_{xx} + \theta ds \tag{2.17}$$

Hence

$$dv = -Pd\eta_{xx} + dv' \tag{2.18}$$

The quantity dv' is an exact differential since dv is exact, and

$$Pd\eta_{xx} = d(P\eta_{xx}) \tag{2.19}$$

The value of v' may be obtained by integrating first along an isothermal path $\theta = 0$ along which $t_{11} = Ce_{xx}$, then along a constant

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deformation path $de_{xx} = 0$ by increasing the temperature to θ . This yields

$$v' = \frac{1}{2}Ce_{xx}^2 + \frac{1}{2}\frac{c\theta^2}{T_r}$$
(2.20)

This is obtained by putting

$$ds = \frac{cd\theta}{T_{\star}} \tag{2.21}$$

along the path $de_{xx} = 0$, hence c is the specific heat per unit volume under the condition of plane strain with $e_{xx} = 0$ while $t_{22} = 0$, i.e. for zero stress in the y direction. The latter condition complies with the Kirkhoff-Bernouilli assumption for thin plate bending.

The value of v becomes

$$v = -\mathbf{P}\eta_{xx} + v' \tag{2.22}$$

or

$$v = -P\eta_{xx} + \frac{1}{2}Ce_{xx}^2 + \frac{1}{2}\frac{c\theta^2}{T_r}$$
(2.23)

Note that C is the isothermal elastic modulus for the deformation with plane strain and $t_{22} = 0$. In classical Elasticity its value is

$$C = \frac{E}{1 - v^2}$$

where E is Young's modulus, and v is Poisson's ratio.

We still need another physical relation which we write

$$T_r s = \gamma e_{xx} + c\theta \tag{2.24}$$

where γ is an experimental physical constant.

The value of θ obtained from (2.24) is substituted in (2.20). Hence

$$v' = \frac{1}{2} \left(C + \frac{\gamma^2}{T_r c} \right) e_{xx}^2 - \frac{\gamma}{c} s e_{xx} + \frac{1}{2} \frac{T_r}{c} s^2$$
(2.25)

From the value (2.17) of dv' we derive

$$t_{11} = \frac{\partial v'}{\partial e_{xx}} = \left(C + \frac{\gamma^2}{T_r c}\right) e_{xx} - \frac{\gamma}{c} s$$

$$\theta = \frac{\partial v'}{\partial s} = -\frac{\gamma}{c} e_{xx} + \frac{T_r}{c} s$$
(2.26)

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Hence $C + \gamma^2/T_r c$ is the adiabatic elastic modulus, (for s = 0). Elimination of s between equations (2.26) also yields

$$t_{11} = \mathrm{C}e_{xx} - \frac{\gamma}{\mathrm{T}_r}\theta \tag{2.27}$$

which gives another physical interpretation of the constant γ in terms of thermal expansion.

Let us examine two extreme cases, one of isothermal deformation $(\theta = 0)$ the other of adiabatic deformation (s = 0). We shall evaluate the value of V for each case.

For $\theta = 0$ the value of V, from, (2.4) and (2.23), is

$$\mathbf{V} = \int_{\mathbf{A}} \left(-\mathbf{P}\eta_{xx} + \frac{1}{2}\mathbf{C}e_{xx}^2 \right) d\mathbf{A}$$
(2.28)

Assuming a sinusoidal deflection

$$w = q_0 \cos \frac{\pi x}{l} \tag{2.29}$$

we derive, from (2.23)

$$\mathbf{V} = \frac{\pi^2}{4l} a q_0^2 (\mathbf{P}_c - \mathbf{P})$$
(2.30)

where

$$P_c = \frac{\pi^2 a^2}{12 l^2} C$$
 (2.31)

This is the classical value of the Euler critical buckling load for isothermal deformations.

For s = 0 i.e. for adiabatic deformation, from (2.4), (2.22) and (2.23),

$$V = \iint_{A} \left[-P\eta_{xx} + \frac{1}{2} \left(C + \frac{\gamma^2}{T_{\mu}c} \right) e_{xx}^2 \right] dA \qquad (2.32)$$

we derive

$$V = \frac{\pi^2}{4l} a q_0^2 (P_a - P)$$
 (2.33)

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where

$$P_{a} = \frac{\pi^{2}}{12} \frac{a^{2}}{l^{2}} \left(C + \frac{\gamma^{2}}{T_{r}c} \right)$$
(2.34)

is the Euler critical buckling load using the adiabatic elastic modulus

$$C + \frac{\gamma^2}{T_r c}$$
.

As soon as $P > P_c$ hence if P is only slightly larger that the isothermal critical load the value of V is a maximum for $q_0 = 0$. Hence in this case according to (2.9) the *entropy* S' is a minimum in the equilibrium state $q_0 = 0$.

Since we have assumed zero mass hence zero kinetic energy, the buckling should be infinitely fast at $P = P_a$.

For $P > P_a$ i.e. for loads superior to the adiabatic buckling load the conservation of energy constraint cannot be verified unless we take into account the kinetic energy of if we add a damping device which damps out the lateral deflection of the plate. In the latter case the energy absorbed by the damper is transformed into heat which is added to the thermal bath thus increasing its entropy.

3. Application of onsager's principle to thermoelastic buckling

We shall consider the rate of entropy production in the total system composed of the plate and thermal bath. Following a procedure introduced many times previously [6-7,11-13] we write the entropy production rate

$$\dot{S}' = -\frac{1}{T_r^2} \int_A \frac{\partial \theta}{\partial y} \dot{H}_y dA + \frac{2}{T_r^2} \int_{-1/2}^{1/2} \theta \dot{H}_a dx \qquad (3.1)$$

where H_y is the heat displacement across the plate thickness, hence in the y direction, and \dot{H}_y is its time derivative. Since we are dealing with a thin plate, the heat flow in other directions may be neglected. In the second integral θ and \dot{H}_a represent the temperature increase and heat flow at the face y = a/2. The second integral is multiplied by two since there are two faces. Heat conduction is expressed by

$$\dot{\mathbf{H}}_{\mathbf{y}} = -k \frac{\partial \theta}{\partial y} \quad \dot{\mathbf{H}}_{a} = \mathbf{K}\theta$$
 (3.2)

where k is the thermal conductivity of the plate and K the boundary heat transfer coefficient. With those relations we write (3.1) as, [6-7,11-12]

$$\dot{S}' = \frac{1}{k} \int_{A} \dot{S}_{y}^{2} dy + \frac{2}{K} \int_{-l/2}^{l/2} \dot{S}_{a}^{2} dx$$
(3.3)

We have used here the entropy displacement defined as

$$S_y = \frac{H_y}{T_r} \quad S_a = \frac{H_a}{T_r} \tag{3.4}$$

This concept was introduced earlier [11]. Actually it is more convenient to work with a dissipation function

$$D = \frac{1}{2}T_{r}\dot{S}' = \frac{T_{r}}{2k}\int_{A}\dot{S}_{y}^{2}dy + \frac{T_{r}}{K}\int_{-l/2}^{l/2}\dot{S}_{a}^{2}dx$$
(3.5)

At this point we must note an important *holonomic* conservation equation. We write

$$h = -\frac{\partial H_y}{\partial y}$$
(3.6)

Dividing by T, it becomes

$$s = -\frac{\partial S_y}{\partial y} \tag{3.7}$$

It should be pointed out that this relation holds for linear perturbations. The thermodynamic state of the system has been defined by writing

$$w(x) = q_0 \cos \frac{\pi x}{l}$$

$$S(x,y) = \cos \frac{\pi x}{l} \sum_{i=1}^{l} q_i \varphi_i(y)$$

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where i = 1, 2, ... etc. and φ_i are suitably chosen functions of y while q_0 and q_i represent a large number of generalized coordinates, theoretically infinite in number. The use of the cos $(\pi x/l)$ factor for the deflection shape is not an approximation since it is easily verified that it is the correct value. Actually it amounts to choosing the first term of a Fourier series whose terms are uncoupled because of orthogonality. With the value s given by (3.7) and (3.8) and the value e_{xx} given by (2.3) and (3.8) expression (2.4) becomes

$$\mathbf{V} = \frac{1}{2}a_{00}q_0^2 + \sum_{i=1}^{i}a_{0i}q_0q_i + \frac{1}{2}\sum_{i=1}^{ij}a_{ij}q_iq_j$$
(3.9)

where

$$a_{00} = \frac{\pi^4}{24} \frac{a^3}{l^3} \left(C + \frac{\gamma^2}{T_r c} \right) - \frac{\pi^2}{2l} Pa$$

$$a_{0i} = -\frac{1}{2c} \frac{\gamma}{c} \frac{\pi^2}{l} \int_{-a/2}^{a/2} y \varphi'_i dy$$

$$a_{ij} = \frac{1}{2} \frac{T_r}{c} l \int_{-a/2}^{a/2} \varphi'_i \varphi'_j dy$$
(3.10)

The prime denotes the derivative of φ_t with respect to y. Similarly the dissipation function (3.5) is expressed as the positive definite quadratic form

$$D = \frac{1}{2} \sum_{ij}^{ij} b_{ij} \dot{q}_i \dot{q}_j \quad (i, j = 1, 2, \dots \text{ etc})$$
(3.11)

As already shown repeatedly [4-6] Onsager's principle [14-17] may be expressed as

$$\frac{\partial \mathbf{V}}{\partial q_l} + \frac{\partial \mathbf{D}}{\partial \dot{q}_l} = 0 \quad (l = 0, 1, 2, \cdots \text{ etc})$$
(3.12)

This assumes that we neglect inertia effects, otherwise as indicated below a kinetic energy term must be added.

Equations (3.12) are the Lagrangian equations of linear thermodynamics [4-6] in the absence of driving forces and mass. The next step is to introduce normal coordinates ξ_s such that

$$\sum_{ij}^{ij} a_{ij} q_i q_j = \sum_{s}^{s} r_s \xi_s^2$$

$$\sum_{j}^{ij} b_{ij} \dot{q}_i \dot{q}_j = \sum_{s}^{s} \dot{\xi}_s^2$$
(3.13)

Note that these quadratic forms are both positive-definite. This is evident for the second. For the first this can be seen since for $q_0 = 0$ we obtain

$$V = \frac{1}{2} \frac{T_r}{c} \int_{A} s^2 ds = \frac{1}{2} \sum_{ij}^{ij} a_{ij} q_i q_j > 0$$
(3.14)

Hence $r_s > 0$. The transformation to normal coordinates is expressed by

$$q_i = \sum_{i=1}^{s} \psi_i^s \xi_s \tag{3.15}$$

In terms of the coordinates q_0 and ξ_s the value of V and D become

$$V = \frac{1}{2}a_{00}q_0^2 + q_0\sum_{i=1}^{si}a_{0i}\psi_i^s\xi_s + \frac{1}{2}\sum_{i=1}^{s}r_s\xi_s^2$$
$$D = \frac{1}{2}\sum_{i=1}^{s}\xi_s^2$$

The Lagrangian equations (3.12) are now

$$\frac{\partial V^2}{\partial q_0} = 0$$
(3.16)
$$\frac{\partial V}{\partial \xi_s} + \frac{\partial D}{\partial \dot{\xi}_s} = 0$$

or

$$a_{00}q_{0} + \sum^{si} a_{0i}\psi^{s}_{i}\xi_{s} = 0$$

$$r_{s}\xi_{s} + \xi_{s} = -\sum^{i} a_{0i}\psi^{s}_{i}q_{0}$$
(3.17)

General solutions are obtained by considering values q_0 and ξ_s proportional to an exponential time factor $\exp(pt)$.

Omitting this factor equations (3.17) become

$$a_{00}q_{0} + \sum_{i}^{si} a_{0i}\psi_{i}^{s}\xi_{s} = 0$$

$$(r_{s} + p)\xi_{s} = -\sum_{i}^{i} a_{0i}\psi_{i}^{s}q_{0}$$
(3.18)

We solve the second set of equations for ξ_s and substitute in the first. After cancelling the factor q_0 we derive

$$a_{00} - \sum_{r=1}^{s} \frac{\beta_s}{p+r_s} = 0$$
 (3.19)

where

$$\beta_s = (\sum_{i=0}^{i} a_{0i} \psi_i^s)^2 > 0 \tag{3.20}$$

Putting

$$\beta_s = \frac{\pi^4 a^3}{24 l^3} \mathbf{B}_s \tag{3.21}$$

and introducing the value (3.10) of a_{00} , equation (3.19) yields

$$P = \frac{\pi^2 a^2}{12 l^2} \left(C + \frac{\gamma^2}{cT_r} - \sum_{r=1}^{s} \frac{B_s}{p+r_s} \right)$$
(3.22)

For p = 0 this must yield the isothermal buckling load. (2.31) hence

$$\frac{\gamma^2}{cT_r} = \sum_{r=1}^{s} \frac{\mathbf{B}_s}{r_s} \tag{3.23}$$

With this value of γ^2/cT_r equation (3.22) is written

$$\mathbf{P} = \frac{\pi^2 a^2}{12 l^2} \left(\mathbf{C} + \sum_{j=1}^{s} \frac{p}{p+r_s} \mathbf{B}_s \right)$$
(3.24)

This equation provides the value of p which varies from zero to infinity when P varies between the isothermal buckling load P_c and the adiabatic buckling load P_a .

The physical significance of expression (3.24) is brought to light by putting $q_0 = 0$ in the last of equations (3.17). They become

$$r_s \xi_s + \dot{\xi}_s = 0 \tag{3.25}$$

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and govern the thermoelastic relaxation modes of the plate. These modes are

$$\xi_s = \mathbf{A}_s e^{-r_s \mathbf{i}}$$

They show how an initial temperature disturbance θ decays to zero. The parameters r_s are the relaxation constants of these modes. They represent *internal coordinates* of the system discussed in earlier work [4]. The quantity

$$\widehat{\mathbf{B}} = \mathbf{C} + \sum_{p=1}^{s} \frac{p}{p+r_s} \mathbf{B}_s$$
(3.26)

may be considered as an operator where p represents symbolically a time derivative p = d/dt.

In this case the meaning of the operator $p/(p + r_s)$ is

$$\frac{p}{p+r_s}f(t) = \int_0^t \exp\left[r_s(\tau-t)\right] df(\tau)$$
(3.27)

For the case of harmonic time dependence, $p = i\omega$ where ω is the circular frequency. The operator \hat{B} is of the general form derived earlier for linear thermodynamic systems with internal coordinates [4-6]. The external coordinate in the present case is the observed deflection. When the observed coordinates are mechanical we have used the appellation viscoelastic operator to refer to an expression such as (3.26) even if the internal coordinates extend beyond mechanical phenomena and include not only thermoelastic effects, but chemical reactions, phase changes, mass transport, electrical effects etc. Equation (3.24) also confirms the rule by which equations for viscoleastic and relaxation phenomena are obtained from the purely elastic relations by replacing the elastic coefficients by their corresponding operators. This was established in 1954-55 in all its generality and referred to as the "viscoelastic correspondance principle". [4,18]. It was applied to the analysis of folding instability of a viscoelastic layer in compression embedded in a viscoelastic medium [19].

It was shown that the thermodynamic restrictions have an important bearing on the physical behavior.

A final remark is in order concerning the physical significance of the buckling model for loads larger than the value for adiabatic M. A. Biot

buckling. In the abscence of kinetic energy internal energy conservation constraints cannot be verified. However the physical reality of the model may be restored by attaching a viscous damping device which damps out the lateral deflection of the plate. This amounts to introducing an additional term in the dissipation formation. We write

$$\mathbf{D} = \frac{1}{2} \sum_{ij}^{ij} b_{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} b_0 \dot{q}_0^2$$
(3.27)

The coefficient b_0 measures the viscous damping of the lateral restraint. Proceeding as above equation (3.24) is replaced by

$$\mathbf{P} = \frac{\pi^2}{12} \frac{a^2}{l^2} \left(\mathbf{C} + \mathbf{B}' p + \sum_{s=1}^{s} \frac{p}{p + r_s} \mathbf{B}_s \right)$$
(3.28)

where

$$\mathbf{B}' = \frac{24l^3}{\pi^4 a^3} b. \tag{3.29}$$

It is verified that the operator in the bracket has the same form (5.20) as below, which is obtained from the general Lagrangian thermodynamics. In this, case due to presence of the term B'p representative of the damping device, the rate of buckling measured by p remains finite at the adiabatic critical load. The energy is dissipated into heat by the damper in the thermal bath thus increasing the total entropy of the isolated system.

Equation (3.28) is of the same type in the case of viscoelastic buckling [19] as to be expected since both cases belong to the same phenomenological Lagrangian theory.

4. SOLUTION DERIVED FROM CLASSICAL FIELD EQUATIONS

The foregoing analysis is based on Lagrangian thermodynamic concepts in the framework of Onsager's principle. It is of interest to compare this general analysis with the results obtained by solving directly the differential equations for the elastic deformation and the temperature field. This was already accomplished previously by using the analogy between thermoelasticity and the theory of porous solids saturated by a viscous fluid [2]. We will show that the same results are obtained from the differential equations of thermoelasticity, however using an entirely different method of solution. For simplicity we shall restrict the analysis to the case where the faces of the plate are thermally insulated.

We consider a cross section of the plate at any abscissa x. From equation (2.24) we derive

$$T_r \frac{\partial s}{\partial t} = \gamma \frac{\partial e_{xx}}{\partial t} + c \frac{\partial \theta}{\partial t}$$
(4.1)

Equations (3.4), (3.7) yield

$$\frac{\partial s}{\partial t} = -\frac{\partial^2 S_y}{\partial t \partial y} = -\frac{1}{T_r} \frac{\partial H_y}{\partial y}$$
(4.2)

and from (3.2) we obtain

$$\frac{\partial s}{\partial t} = \frac{k}{T_r} \frac{\partial^2 \theta}{\partial y^2}$$
(4.3)

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Also from (2.3) we may write

$$\frac{\partial e_{xx}}{\partial t} = -y \frac{\partial^3 w}{\partial t \partial x^2}$$
(4.4)

Hence the differential equation (4.1) becomes

$$k\frac{\partial^2\theta}{\partial y^2} = -\gamma y \frac{\partial^3 w}{\partial t \partial x^2} + c\frac{\partial \theta}{\partial t}$$
(4.5)

We assume an exponential time dependence where the variable w and θ are proportional to exp(pt). Omitting the time factor, equation (4.5) is written

$$k\frac{\partial^2\theta}{\partial y^2} = -p\gamma y\frac{d^2w}{dx^2} + pc\theta$$
(4.6)

This equation governs the distribution of θ along y hence across the thickness at my given abscissa assuming the curvature dw^2/dx^2 to be a known function of location.

Since we assume the faces to be thermally insulated the boundary conditions are

$$\frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad y = \pm a \tag{4.7}$$

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The solution of the differential equation (4.6) satisfying these boundary conditions is

$$\theta = \frac{\gamma d^2 w}{c dx^2} \left\{ y - \frac{\sinh \alpha y}{\alpha \cosh\left(\frac{\alpha a}{2}\right)} \right\}$$
(4.8)

with

$$\alpha^2 = \frac{pc}{k}$$

The bending moment at the absicssa x is

$$\mathcal{M} = \int_{-\frac{a}{2}}^{\frac{a}{2}} y t_{11} dy$$
 (4.9)

From (2.27), (2.3) we obtain

$$t_{11} = Ce_{xx} - \frac{\gamma}{T_r}\theta = -Cy\frac{d^2w}{dx^2} - \frac{\gamma}{T_r}\theta \qquad (4.10)$$

Substitution of t_{11} into the integral (4.9) with the value (4.8) for θ yields the bending moment in the form

$$\mathscr{M} = -\frac{a^3}{12}\hat{B}\frac{d^2w}{dx^2}$$
(4.11)

with

$$\hat{\mathbf{B}} = \mathbf{C} + \frac{\gamma^2}{cT_r} \left[1 - \frac{3}{z^2} \left(1 - \frac{\tanh z}{z} \right) \right]$$

$$z = \frac{\alpha a}{2} = \frac{a}{2} \sqrt{\frac{pc}{k}}$$
(4.12)

For comparison with previous results we expand the bracket in partial fractions. We use a classical expansion of the meromorphic function

$$\tanh z = \sum_{s=0}^{\infty} \frac{2z}{z^2 + \left(s + \frac{1}{2}\right)^2 \pi^2}$$
(4.13)
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We first note

$$1 = \lim_{z \to 0} \frac{\tanh z}{z} = \sum_{0}^{\infty} \frac{1}{\left(s + \frac{1}{2}\right)^2 \pi^2}$$
(4.14)

Hence we obtain

$$1 - \frac{\tanh z}{z} = 2\sum_{s=0}^{\infty} \frac{z^2}{z^2 + \left(s + \frac{1}{2}\right)^2 \pi^2}$$
(4.15)

Furthermore

$$\frac{3}{z^2} \left(1 - \frac{\tanh z}{z} \right) = 6 \sum_{s=0}^{\infty} \frac{1}{z^2 + \left(s + \frac{1}{2}\right)^2 \pi^2}$$
(4.16)

Also

$$1 = \lim_{z \to 0} \frac{3}{z^2} \left(1 - \frac{\tanh z}{z} \right) = 6 \sum_{s=0}^{\infty} \frac{1}{\left(s + \frac{1}{2}\right)^2 \pi^2}$$
(4.17)

Hence

$$1 - \frac{3}{z^2} \left(1 - \frac{\tanh z}{z} \right) = 6 \sum_{s=0}^{\infty} \frac{z^2}{\left(s + \frac{1}{2} \right)^4 \left[z^2 + \left(s + \frac{1}{2} \right)^2 \pi^2 \right]}$$
(4.18)

This expansion is now substituted in the value (4.12) of \hat{B} using expression (4.12) for z. We obtain

$$\hat{\mathbf{B}} = \mathbf{C} + \sum_{0}^{\infty} \frac{\mathbf{B}_{s} p}{p + r_{s}}$$
(4.19)

where

$$B_{s} = 96 \frac{\gamma^{2}}{cT_{r}(2s+1)^{4}\pi^{4}}$$
(4.20)
$$r_{s} = (2s+1)^{2}\pi^{2} \frac{k}{ca^{2}}$$

This result coincides with the one derived previously by an entirely different method [2]. However we must take care of the following M. A. Biot

remark. In the present analysis c represents the specific heat for plane strain and zero stress $t_{22} = 0$ in the y direction. However in reference [2] we have used c to represent the heat capacity at constant volume. The two results check if we take this difference into account.

Note that the form of the operator (4.19) for \hat{B} is the same as in the general case (3.26).

We may now go back to the buckling problem using the classical relation between bending moment and deflection

$$\mathcal{M} = \mathbf{P}aw \tag{4.21}$$

Hence with the value (4.11) of \mathcal{M}

$$\frac{a^3}{12}\hat{B}\frac{d^2w}{dx^2} + Paw = 0$$
(4.22)

A solution $w = q_0 \cos \pi x/l$ yields

$$\mathbf{P} = \frac{\pi^2 a^2}{12l^2} \,\hat{\mathbf{B}} \tag{4.23}$$

which coincides with the value (3.24) for the more general case.

5. Dynamic buckling

In the foregoing analysis we have neglected the inertia force. A general Lagrangian thermodynamics taking inertia forces into account was developed in [3,6,18].

The Lagrangian equations for the generalized coordinates q_l are

$$\frac{d}{dt}\frac{\partial\mathcal{F}}{\partial\dot{q}_{i}} + \frac{\partial D}{\partial\dot{q}_{i}} + \frac{\partial\mathcal{P}}{\partial q_{i}} = Q_{i}$$
(5.1)

where

$$\mathscr{T} = \frac{1}{2} \sum_{k=1}^{lk} m_{lk} \dot{q}_{l} \dot{q}_{k}$$
(5.2)

is the kinetic energy,

$$\mathbf{D} = \frac{1}{2} \sum_{ij}^{lk} b_{ij} \dot{q}_i \dot{q}_j \tag{5.3}$$

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is the general dissipation function derived from the rate of entropy production and

$$\mathscr{P} = \frac{1}{2} \sum_{k=1}^{lk} a_{lk} q_l q_k \tag{5.4}$$

is a mixed non-classical mechanical and thermodynamic potential. It is expressed as

$$\mathscr{P} = \mathbf{U} - \mathbf{T}_r \mathbf{S} + \boldsymbol{\phi} \tag{5.5}$$

where ϕ represents the mechanical force potential.

The generalized forces Q_l are driving forces applied to the system and defined in such a way that $\sum_{l}^{l} Q_l \delta q_l$ is their virtual work. They also include purely thermodynamic thermal forces due to driving temperatures and defined in a similar way [6,11-12].

Instability without driving forces is governed by the equations

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{q}_{l}} + \frac{\partial\mathbf{D}}{\partial\dot{q}_{l}} + \frac{\partial\mathcal{P}}{\partial q_{l}} = 0$$
(5.6)

Instability can only occur if \mathcal{P} may become negative as in the present example of plate buckling. Inertia forces are taken into account by adding the kinetic energy of the plate.

$$\mathcal{F} = \frac{1}{2}m\dot{q}_0^2 \tag{5.7}$$

Assuming a sinusoidal deflection, the value of m is

$$m = \frac{1}{2}\rho a l \tag{5.8}$$

where ρ is the plate mass density.

With the value (5.7) of the kinetic energy and putting $\phi = 0$ for the force potential we solve the dynamic stability equation for the plate following the same procedure as before. Equation (3.19) becomes

$$a_{00} - \sum_{r=1}^{s} \frac{\beta_s}{p+r_s} + mp^2 = 0$$
 (5.9)

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and equation (3.24) for the compressive load P is replaced by

$$\mathbf{P} = \frac{\pi^2 a^2}{12l^2} \left(\mathbf{C} + \sum_{p=1}^{s} \frac{p}{p+r_s} \mathbf{B}_s + \frac{12\rho l^4}{\pi^4 a^2} p^2 \right)$$
(5.10)

In this case the rate of deflection measured by p remains finite as the compressive load P increases. It should be noted that this requires a more detailed analysis since modes of buckling

$$w = q_0 \cos n \frac{\pi}{l} x$$

representing several half waves may have a higher value of the rate p for a given load P (see [3] page 335).

It is interesting to note that the problem of thermoelastic buckling treated in the foregoing analysis is a particular case of a much more general treatment of unstable thermodynamic systems. By distinguishing between two sets of coordinates one called *external* the other called *internal* [3-4,6]. We shall write the mixed potential \mathcal{P} pas a sum of two terms

$$\mathscr{P} = \mathscr{P}_{ex} + \mathscr{P}_{in} \tag{5.11}$$

where

$$\mathscr{P}_{ex} = \mathscr{P}(q_l) = \frac{1}{2} \sum_{k=1}^{lk} a_{lk} q_l q_k.$$
 (5.12)

is a function of external coordinates q_l only and is not positive definite. The other term

$$\mathscr{P}_{in} = \mathscr{P}(q_l, q_\mu) \tag{5.13}$$

is a function of both external coordinates q_i and internal coordinates q_u and is positive definite. The dissipation function

$$\mathbf{D} = \mathbf{D}(\dot{q}_l, \dot{q}_\mu) \tag{5.14}$$

is a function of both external and internal coordinates, and the kinetic energy

$$\mathcal{T} = \frac{1}{2} \sum_{k=1}^{lk} m_{lk} \dot{q}_l \dot{q}_k \tag{5.15}$$

is a function of the external coordinates q_i only.

The Lagrangian equations are

$$\frac{d}{dt}\frac{\partial \mathcal{F}}{\partial \dot{q}_{l}} + \frac{\partial \mathcal{P}_{ex}}{\partial q_{l}} + \frac{\partial \mathcal{P}_{in}}{\partial q_{l}} + \frac{\partial \mathbf{D}}{\partial \dot{q}_{l}} = \mathbf{Q}_{l}$$
$$\frac{\partial \mathcal{P}_{in}}{\partial q_{\mu}} + \frac{\partial \mathbf{D}}{\partial \dot{q}_{\mu}} = 0$$
(5.16)

By definition the internal coordinates are not subject to driving forces hence $Q_{\mu} = 0$.

Let us put

$$\mathbf{Q}_{i}^{\prime} = \mathbf{Q}_{i} - \frac{d \,\partial \mathcal{F}}{dt \,\partial \dot{q}_{i}} - \frac{\partial \mathcal{P}_{ex}}{\partial q_{i}} \tag{5.17}$$

Hence equations (5.16) become

$$\frac{\partial \mathscr{P}_{in}}{\partial q_{i}} + \frac{\partial \mathbf{D}}{\partial \dot{q}_{i}} = \mathbf{Q}'_{i}$$

$$\frac{\partial \mathscr{P}_{in}}{\partial q_{\mu}} + \frac{\partial \mathbf{D}}{\partial \dot{q}_{\mu}} = 0$$
(5.18)

In 1954 [4] and subsequent work [12] we derived a general impedance expression for the response of the thermodynamic system represented by equations (5.18). This expression is

$$\mathbf{Q}_{l}^{\prime} = \sum_{k}^{k} \widehat{\mathbf{Z}}_{lk} q_{k} \tag{5.19}$$

with

$$\hat{Z}_{kl} = \hat{Z}_{lk} = \sum_{k=1}^{s} Z_{lk}^{(s)} \frac{p}{p+r_s} + Z_{lk} + p Z_{lk}^{\prime}$$
(5.20)

where Z_{lk} , Z'_{lk} and $Z'_{lk}^{(s)}$ are all non negative and $r_s > 0$. The parameters r_s are relaxation constants of the internal coordinate system. The quantity p is either a differential operator p = d/dt or represents the coefficient in an exponential time dependence proportional to $\exp(pt)$ where p is either real, complex, or equal to $p=i\omega$ for harmonic driving forces. Using the operational notation equations (5.19) may be written

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$$\sum_{i=1}^{k} (\hat{Z}_{lk} + a_{lk} + p^2 m_{lk}) q_k = Q_l$$
 (5.21)

This relates the response of the system to external driving forces Q_i . If we put $Q_i = 0$ equations (5.21) become homogeneous

$$\sum^{k} (\hat{Z}_{lk} + a_{lk} + p^2 m_{lk}) q_k = 0$$
 (5.22)

Cancelling the determinant provides the damped oscillations of the system or the exponential unstable modes. As will be shown below the latter are always non-oscillatory corresponding to values of p which are real and positive.

6. THEOREM ON THE NON-OSCILLATORY CHARACTER OF INTABILITY

In the foregoing analysis we have assumed that $\exp(pt)$ is a real increasing exponential i.e. that the instability is non-oscillatory. This assumption is justified by a general theorem concerning the general dynamic instability equations (5.6). The theorem was derived in the author's book ([3] page 442) which states that the unstable solutions of those equation are proportional to a real exponential function. For the convenience of the reader this proof will be briefly summarized.

With q_1 proportional to exp(pt) equations (5.6) are written

$$p^{2} \sum_{k}^{k} m_{lk} q_{k} + p \sum_{k}^{k} b_{lk} q_{k} + \sum_{k}^{k} a_{lk} q_{k} = 0$$
 (6.1)

For any solution p, q_k which is not real there is a conjugate one p_k^* , q_k^* such that

$$p^{*2} \sum_{k=1}^{k} m_{lk} q_{k}^{*} + p^{*} \sum_{k=1}^{k} b_{lk} q_{k}^{*} + \sum_{k=1}^{k} a_{lk} q_{k}^{*} = 0$$
(6.2)

Equations (6.1) are multiplied by q_k^* and summed, similarly equations (6.2) are multiplied by q_l and summed. The two results are subtracted from each other. Taking into account the symmetry properties $m_{lk} = m_{kl}$, $b_{lk} = b_{kl}$, $a_{lk} = a_{kl}$ we obtain

$$(p - p^*)[(p + p^*)m_{lk}q_lq_k^* + b_{lk}q_lq_k^*] = 0$$
(6.3)

The kinetic energy and dissipation function are positive definite hence

$$\sum_{k=1}^{lk} m_{lk} q_{l} q_{k}^{*} > 0$$

$$\sum_{k=1}^{lk} b_{lk} q_{l} q_{k}^{*} > 0$$
(6.4)

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Moreover instability requires

$$p + p^* > 0$$
 (6.5)

As a consequence equation (6.3) implies

$$p = p^* \tag{6.6}$$

Hence p is real and the instability is non oscillatory.

Note that this theorem is very general and applies to all linear unstable thermodynamic systems governed by Onsager's principle with a potential \mathcal{P} .

7. VALIDITY EXTENDED TO INSTABILITY WITH MOLECULAR DIFFUSION

The analysis of thermodynamic instability which is presented here in the context of thermal diffusion remains valid for the much more general case of unstable systems coupled with molecular diffusion of chemical species since they are governed by the same phenomenological Lagrangian equations of irreversible processes. For instance in the case of buckling, the bending deformation induces differences in chemical potentials of the molecular constituents between the regions in tension and compression, thus generating a diffusion and a creep buckling, governed by the general equation (5.6). This is also implicit in the physical meaning of viscoelasticity considered in the thermodynamic theory of initially stressed solids [3]. Similarly as shown earlier [1-2,11] the theory of thermoelasticity is in complete analogy with the theory porous solids, where a viscous fluid is allowed to diffuse through the pores. In this case the fluid pressure is the analogue of the temperature and the fluid flow corresponds to heat flow.

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