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# EXTRAIT

A virtual dissipation principle and Lagrangian equations in non-linear irreversible thermodynamics

by M.A. BIOT



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## A virtual dissipation principle and Lagrangian equations in non-linear irreversible thermodynamics

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Abstract. — A principle of virtual dissipation is derived which generalizes d'Alembert's principle to non-linear irreversible thermodynamics, leading to a corresponding Hamiltonian form and Lagrangian equations. It constitutes the fundamental mathematical tool for the analysis of general non-linear dissipative physical systems by methods similar to those of classical mechanics. The principle provides a derivation of the differential field equations or the Lagrangian equations which govern any particular dissipative thermodynamic system and is not based on a priori knowledge of the field equations. Results are essentially an ultimate development of the author's earlier work originated in 1954 which was presented in a more restricted context.

#### 1. INTRODUCTION

A general thermodynamics of irreversible processes based on Lagrangian concepts and equations with generalized coordinates was initiated around 1954 [1-3], and developed mainly in the context of viscoelasticity and linear processes. That the essential concepts and procedures are applicable to a wider range of phenomena including non-linearity has been shown by the treatment of special cases, such as non-linear heat transfer [5] non-linear thermoelasticity [4] nonlinear porous solids [8-9] and non-linear viscoelasticity [10]. Current litterature on the subject seems to reflect a lack of awareness of the extensive generality and potentiality of the aforementioned material which represents a distinct school of thought in the field of nonequilibrium thermodynamics.

Our purpose is to derive what is referred to here as a principle of virtual dissipation which constitutes a generalized form of d'Alembert's principle applicable to non-linear irreversible thermodynamics. This provides a unified treatment of a vast category of non-linear dissipative systems, leading, as in classical mechanics, to Lagrangian type equations. Fundamentally the generalization to thermodynamics is obtained by adding reversed dissipative forces to the reversed inertia forces of the classical d'Alembert's principle. A distinctive feature of this variational principle is due to the fact that its formulation does not require any knowledge of the differential field equations. On the contrary the differential equations or the Lagrangian equations of the system may be derived directly from the principle itself.

The formulation uses a thermostatic expression referred to here as a "collective potential". Its use as a new thermodynamic potential was already introduced in much earlier work of 1954 [1-2]. Systems which are holonomic or approximately holonomic from the thermodynamic viewpoint as well as quasi-reversible obey Lagrangian equations with a dissipation function. The latter embodies non-linearity in the sense that the corresponding Onsager coefficients depend in the instantaneous values of the state variables. A corollary is a theorem of instantaneous minimum entropy production already enounced earlier in a more restricted context (1955) [2].

Instability of thermodynamic equilibrium in a state of minimum entropy as well as the non-oscillatory character of this instability was also discussed earlier [6-7, 4]. It is briefly recalled here. The analysis brings out, an important *distinction between the concepts of entropy production and energy dissipation*. The latter has meaning only if the entropy production is associated with a thermal reservoir at a given temperature. As a consequence two types of dissipations are brought out and referred to as *relative* and *intrinsic* dissipation.

As an illustration, two applications are derived to non-linear thermoelasticity and to non-linear heat conduction. In the latter case it leads to the basic variational principle for heat conduction used extensively in the author's book [5]. The application to thermoelasticity also illustrates the distinction between the concept of local and collective dissipation associated with the same entropy production.

#### 2. COLLECTIVE POTENTIAL

It was shown originally in 1954 [1] and in some subsequent work [2-3] that a general Lagrangian thermodynamics applicable to a large category of irreversible processes can be developed by introducing a fundamental *thermodynamic potential* different from the classical ones, written as

$$\mathbf{V} = \mathbf{U} - \mathbf{T}_r \mathbf{S} \tag{2.1}$$

where U and S are the internal energy and entropy of the system and  $T_r$  is the temperature of a large isothermal reservoir which may exchange thermal energy with the system. We shall refer to V as the *collective potential* and the isothermal reservoir as the *thermal well*. The collective potential was used to describe a system which is not in thermodynamic equilibrium. In particular the temperatures may vary from point to point. While this concept was used mainly in the context of linear thermodynamics its applicability is much more general.

An important property of the collective potential is its *additive* character. If the system is composed of a large number of interacting cells, we define a *cell potential* as

$$\mathscr{V}_{k} = \mathscr{U}_{k} - \mathrm{T}_{r}\mathscr{S}_{k} \tag{2.2}$$

where  $\mathcal{U}_k$  and  $\mathcal{S}_k$  are the internal energy and entropy of the  $k^{th}$  cell. The collective potential is then

$$\mathbf{V} = \sum_{k=1}^{k} \mathscr{V}_{k} = \sum_{k=1}^{k} \mathscr{U}_{k} - \mathbf{T}_{r} \sum_{k=1}^{k} \mathscr{S}_{k}$$
(2.3)

with

$$\mathbf{U} = \sum_{k=1}^{k} \mathscr{U}_{k} \qquad \mathbf{S} = \sum_{k=1}^{k} \mathscr{S}_{k} \tag{2.4}$$

the total internal energy and entropy of the system.

For continuous systems the cells may be infinitesimal and the summations are replaced by volume integrals.

If the cell is sufficiently small to be considered at uniform temperature  $T_k$  and completely described by its entropy  $\mathscr{S}_k$  and any number of independant state variables  $q_i^k$ . We may write

$$d\mathscr{V}_{k} = \sum_{l=1}^{l} \frac{\partial \mathscr{U}_{k}}{\partial q_{l}^{k}} dq_{l}^{k} + \frac{\partial \mathscr{U}_{k}}{\partial \mathscr{S}_{k}} d\mathscr{S}_{k} - \mathbf{T}_{r} d\mathscr{S}_{k}$$
(2.5)

Using the classical relation

$$\frac{\partial \mathcal{U}_k}{\partial \mathcal{S}_k} = \mathbf{T}_k \tag{2.6}$$

we derive

$$d\mathscr{V}_{k} = \sum_{l=1}^{l} \frac{\partial \mathscr{U}_{k}}{\partial q_{l}^{k}} dq_{l}^{k} + \theta_{k} d\mathscr{S}_{k}$$
(2.7)

where

$$\theta_k = \mathbf{T}_k - \mathbf{T}_r = \frac{\partial \mathscr{V}_k}{\partial \mathscr{S}_k} \tag{2.8}$$

is the excess temperature of the cell above the thermal well temperature. Relation (2.7) expressing that the right side is an exact differential plays an important role in applications.

Finally the most significant property of the collective potential is obtained by applying the principle of conservation of energy. We write

$$d\mathbf{U} = d\mathbf{W}_t - d\mathbf{H}_r. \tag{2.9}$$

The heat energy extracted from the system and absorbed by the thermal well is denoted by  $H_r$ , while  $dW_t$  is the work of the external forces applied to the system. Note that by d'Alembert's principle this includes the work of the reversed inertia forces considered as external body forces acting on the system. The entropy of the total combined system *including the thermal well* is

$$S' = S + \frac{H_r}{T_r}$$
(2.10)

Elimination of  $H_r$  and U between equations (2.1) (2.9) and (2.10) yields.

$$d\mathbf{V} - d\mathbf{W}_t = -\mathbf{T}_t d\mathbf{S}'. \tag{2.11}$$

This relation is valid at any particular instant, for arbitrary differentials of the variables defining the system, and for arbitrary transformations whether reversible or irreversible. The rate of entropy production in the total system is  $\dot{S}' = dS'/dt$ . We shall call  $\dot{S}T$ , the power dissipated. It constitutes a measure of the irreversibility.

#### 3. VIRTUAL DISSIPATION PRINCIPLE

A GENERALIZATION OF D'ALEMBERT'S PRINCIPLE TO DISSIPATIVE SYSTEMS

A unified principle analogous to d'Alembert's principle may be derived for combined mechanical and thermodynamic systems. This was already shown in the particular case of thermoelasticity [4]. Actually the procedure illustrated in that particular case is quite general.

The total system is described by generalized coordinates  $q_i$  whose nature may be mechanical, thermodynamic, electrical, chemical or of any other type. Instantaneous transformations are described by generalized velocities  $\dot{q}_i$ .

We define generalized dissipative forces  $X'_i$  by writing,

$$T_r dS' = \sum_{i=1}^{i} X'_i dq_i \tag{3.1}$$

With this definition relation (2.11) becomes

$$d\mathbf{V} - d\mathbf{W}_t + \sum_{i=1}^{i} \mathbf{X}'_i dq_i = 0$$
(3.2)

This relation is valid for arbitrary differentials hence also for arbitrary virtual variations  $\delta q_i$ .

$$\delta \mathbf{V} - \delta \mathbf{W}_t + \sum_{i=1}^{i} \mathbf{X}'_i \delta q_i = 0$$
(3.3)

The term  $\delta W_t$  may be split into two groups

$$\delta \mathbf{W}_{t} = -\sum_{i}^{i} \mathbf{I}_{i} \delta q_{i} + \sum_{i}^{i} \mathbf{Q}_{i} \delta q_{i}$$
(3.4)

where  $-I_i$  are the reversed inertia forces and  $Q_i$  the external forces. Relation (3.3) becomes:

$$\delta \mathbf{V} + \sum_{i}^{i} \mathbf{I}_{i} \delta q_{i} - \sum_{i}^{i} \mathbf{Q}_{i} \delta q_{i} + \sum_{i}^{i} \mathbf{X}_{i}' \delta q_{i} = 0$$
(3.5)

It constitutes the announced *principle of virtual dissipation* and generalizes d'Alembert's principle. The term

$$T_r \delta S' = \sum_{i=1}^{i} X'_i \delta q_i \tag{3.6}$$

represents the virtual dissipation.

-- 10 ---

A Hamiltonian form of this variational principle is of course obtainable by integrating equation (3.7) with respect to time t,

$$\int \left[\delta \mathbf{V} + \sum_{i}^{i} \mathbf{I}_{i} \delta q_{i} - \sum_{i}^{i} \mathbf{Q}_{i} \delta q_{i} + \sum_{i}^{i} \mathbf{X}_{i}' \delta q_{i}\right] dt = 0$$
(3.7)

However nothing essentially new is added by doing this, except for the fact that it provides a direct variational procedure for the evaluation of the inertia term  $I_i$  by means of the kinetic energy according to equation (8.3) below.

In addition to the mechanical constraints the variation  $\delta q_i$  are assumed to obey conservation constraints for energy, mass, electrical charges. As can be seen the reversed dissipative forces -X' play the same role as the reversed inertia forces  $-I_i$  Note that both are *frozen instantaneous values* which remain fixed, while variations are applied.

Virtual thermodynamic equilibrium. The general principle (3.5) may be considered as expressing that the system is in "virtual" thermodynamic equilibrium if we apply reversed dissipative forces  $-X'_i$ . To show this we write equation (3.5) in the form

$$\delta \mathbf{V} + \sum_{i}^{i} \mathbf{I}_{i} \delta q_{i} - \sum_{i}^{i} \mathbf{Q}_{i}' \delta q_{i} = -\mathbf{T}_{r} \delta \mathbf{S}'$$
(3.8)

putting at the same time

$$\mathbf{Q}'_i = \mathbf{Q}_i - \mathbf{X}'_i \qquad \delta \mathbf{S}' = 0 \tag{3.9}$$

This amounts to the statement that by applying reversed dissipative forces  $-X'_i$  the virtual change of entropy  $\delta S'$  of the total system vanishes.

Disequilibrium forces. In systems for which  $X'_i = 0$  no entropy is produced. Such a system may be considered in thermodynamic equilibrium in the sense that virtual changes of entropy vanish. Hence the dissipative forces  $X'_i$  constitute a measure of the departure from equilibrium.

They may be referred to as *disequilibrium forces* a term already introduced earlier [2-3]. Hence some of the instantaneous velocities  $\dot{q}_i$  of the system are due essentially to the disequilibrium forces  $X'_i$ . A complete physical description of the irreversible process requires the knowledge of the functional relations between  $\dot{q}_i$  and  $X_i$ .

$$\mathbf{X}_i' = \mathbf{R}_i'(q_k, \dot{q}_k) \tag{3.10}$$

-- 11 ---

These *rate equations* may be obtained from experimental data or statistical theories. For example they may correspond to viscous forces, reaction or diffusion rates etc. Since R' represents the actual value of  $X'_i$  in the physical process, it obeys the fundamental property

$$\sum_{i=1}^{i} \mathbf{R}_i' \dot{q}_i > 0 \tag{3.11}$$

expressing that the dissipated power is positive definite. It is assumed of course that in the inequality (3.1)  $R'_i$  is not a frozen value but depends on  $\dot{q}_i$ .

Restoring forces. The variation  $\delta V$  is obtained by applying variations  $\delta q_i$  of the state variables in the vicinity of a given configuration  $q_i$ . We may write

$$\delta \mathbf{V} = \sum_{i}^{i} \mathbf{F}_{i}^{\prime} \delta q_{i} \tag{3.12}$$

The coefficients  $F'_i$  of this variational differential form depend on the instantaneous configuration  $q_i$ 

$$\mathbf{F}'_i = \mathbf{F}'_i(q_l) \tag{3.13}$$

The virtual dissipation principle (3.5) becomes

$$\sum_{i}^{i} (\mathbf{F}_{i}' + \mathbf{I}_{i} - \mathbf{Q}_{i} + \mathbf{X}_{i}') \delta q_{i} = 0$$
(3.14)

Hence with  $X'_i = R'_i$  we derive

$$F'_i + I_i - Q_i + R'_i = 0 (3.15)$$

These differential equations govern the evolution of the system. We shall refer to  $F'_i$  as the *restoring forces*. This is justified by the fact that in the absence of kinetic energy and externally applied forces  $(dW_t = 0)$  equation (2.11) becomes

$$\dot{\mathbf{V}} = \sum_{i}^{i} \mathbf{F}_{i}^{i} \dot{q}_{i} = -\mathbf{T}_{r} \dot{\mathbf{S}}^{i}$$
(3.16)

Since  $S'_i > 0$  the value of V decreases with time and for a stable system  $F'_i$  is similar to an elastic restoring force. For an unstable system they will be negative.

— 12 —

#### 4. DISTINCTION BETWEEN RELATIVE AND INTRINSIC DISSIPATION

The dissipation rate  $T_r$ 'S expressed by equation (3.1) may be called "relative dissipation" because it is associated with a thermal well temperature  $T_r$  which may be chosen arbitrarily. The relative character of the dissipation thus defined is easily understood by considering for example the heat produced by friction. The energy is not entirely lost if the heat is produced at a temperature T which is higher than the temperature  $T_r$  of the thermal well. Part of the heat may still be converted into useful work by a Carnot cycle between temperatures T and  $T_r$ .

In general consider a system made up of cells at different temperatures. Consider the  $k^{th}$  cell at the temperature  $T_k$ . The rate of entropy production  $\dot{s}_k^*$  in the cell is associated with two types of dissipation. One type is defined as  $T_k \dot{s}_k^*$  associated with the temperature  $T_k$  of the cell. We shall call it the *intrinsic dissipation* because it represents the energy which is totally lost and cannot be recuperated into useful work unless there is available a thermal well at a temperature  $T_r < T_k$ . On the other hand  $T_r \dot{s}_k^*$  is the *relative dissipation*. It is smaller than  $T_k \dot{s}_k^*$  since part of the heat may still be transformed into useful work in the presence of the thermal well. The difference between the two dissipation rates is

$$(\mathbf{T}_k - \mathbf{T}_r)\dot{s}_k^* = \theta_k \dot{s}_k^* = \theta_k \frac{\dot{h}_k}{\mathbf{T}_k}$$
(4.1)

If  $h_k$  represents the rate of energy transformed into heat,  $\theta_k \dot{h}_k/T_k$  is the amount available for useful work and  $\theta_k/T_k$  is the Carnot cycle efficiency.

We may also define an intrinsic dissipation rate for the collective system as

$$\sum_{k=1}^{k} T_k \dot{s}_k^* = \sum_{i=1}^{i} X_i \dot{q}_i \tag{4.2}$$

and a corresponding virtual intrinsic dissipation

$$\sum_{k=1}^{k} T_k \delta s_k^* = \sum_{i=1}^{i} X_i \delta q_i \tag{4.3}$$

In these expression  $X_i$  are the intrinsic dissipative forces.

#### 5. VIRTUAL INTRINSIC DISSIPATION PRINCIPLE

The variational principle (3.5) may be formulated in terms of intrinsic dissipation. This form is particularly suitable when the system is constituted by a collection of cells each at its own temperature  $T_{\nu}$ . The collective potential is

$$\mathbf{V} = \sum_{k=1}^{k} \mathscr{V}_{k} \tag{5.1}$$

where  $\mathscr{V}_k$  is determined by a number of local cell variables including its entropy  $\mathscr{S}_k$ . This entropy is the sum of two terms

$$\mathscr{S}_k = s_k + s_k^* \tag{5.2}$$

where  $s_k$  is the entropy supplied to the cell and  $s_k^*$  is the entropy produced in the cell.

The variation  $\delta V$  may be written in the form (applying 2.8)

$$\delta \mathbf{V} = \delta_{\mathbf{R}} \mathbf{V} + \sum_{k=1}^{k} \frac{\partial \mathscr{V}}{\partial \mathscr{S}_{k}} \delta s_{k}^{*} = \delta_{\mathbf{R}} \mathbf{V} + \sum_{k=1}^{k} \theta_{k} \delta s_{k}^{*}$$
(5.3)

where  $\delta_R$  is a restricted variation obtained by excluding variations due to  $\delta s_k^*$  and  $\theta_k = T_k - T_r$  is the excess temperature of a cell above that of the thermal well.

We note that for a system of cells the relative virtual dissipation (3.6) is

$$\sum_{i}^{i} X_{i}^{\prime} \delta q_{i} = \mathbf{T}_{r} \delta \mathbf{S}^{\prime} = \mathbf{T}_{r} \sum_{i}^{k} \delta s_{k}^{*}$$
(5.4)

Substitution of the values (5.3) and (5.4) for  $\delta V$  and  $\sum X'_i \delta q_i$  in the variational principle (3.5) yields

$$\delta_{\mathsf{R}}\mathsf{V} + \sum^{i} \mathbf{I}_{i}\delta q_{i} - \sum^{i} \mathbf{Q}_{i}\delta q_{i} + \sum^{k} \mathbf{T}_{k}\delta s_{k}^{*} = 0$$
(5.5)

This provides an alternative form of the variational principle (3.5) in terms of the intrinsic dissipation  $\sum_{k=1}^{i} T_k \delta s_k^*$ , using a restricted form of the variation  $\delta_R V$  which excludes  $\delta s_k^*$ . Thus (5.5) represents a *principle of virtual intrinsic dissipation*. Using expression (4.3) it may be written

$$\delta_{\mathbf{R}}\mathbf{V} + \sum_{i}^{i}\mathbf{I}_{i}\delta q_{i} - \sum_{i}^{i}\mathbf{Q}_{i}\delta q_{i} + \sum_{i}^{i}\mathbf{X}_{i}\delta q_{i} = 0$$
(5.6)

where  $X_i$  are intrinsic dissipative forces. Integration of the variational equation (5.6) with respect to time yields a Hamiltonian form analogous to (3.7).

We may of course write rate equations similar to (3.10) such as

$$X_i = R_i(q_k, \dot{q}_k) \text{ with } \sum_{i=1}^{i} R_i \dot{q}_i > 0$$
(5.7)

Also we may define restricted restoring forces  $F_i$  analogous to  $F'_i$  by the relation

$$\delta_{\rm R} {\rm V} = \sum_{i}^{i} {\rm F}_i \delta q_i \tag{5.8}$$

They are also functions of the instantaneous configuration, hence

$$\mathbf{F}_i = \mathbf{F}_i(q_i) \tag{5.9}$$

The variational principle (5.6) leads to differential equations of evolution of the system in the form

$$F_i + I_i - Q_i + R_i = 0 (5.10)$$

They are analogous to (3.16).

#### 6. ISOTHERMAL AND QUASI-ISOTHERMAL SYSTEMS

In many cases it is possible to assume as a valid approximation completely *isothermal transformations*. In this case the temperature remains constant and uniform throughout, and equal to the temperature T<sub>r</sub> of the thermal well.

Formulation of the virtual dissipation principle in this case is considerably simplified. First we should point out that for such isothermal systems the collective potential (2.1) coincides with the *Helmholtz free energy*.

All cells remaining at constant temperature we put  $T_k = T_r$ . Hence the virtual dissipation is

$$\sum_{k=1}^{k} T_k \delta s_k^* = T_r \sum_{k=1}^{k} \delta s_k^* = T_r \delta S'$$
(6.1)

In this case there is *no distinction between relative and intrinsic dissipation*. Moreover applying equation (2.7) the variation of the collective potential may be written

$$\delta \mathbf{V} = \sum_{k=1}^{k} \delta_{s} \mathscr{V}_{k} + \sum_{k=1}^{k} \theta_{k} \delta \mathscr{S}_{k}$$
(6.2)

- 15 -

where  $\delta_s$  is the variation obtained without varying the cell entropy  $\mathscr{S}_k$  as an independent variable. However since  $T_k = T_r$  is given,  $\mathscr{S}_k$  still varies as a function of other variables. Since  $\theta_k = 0$  we write

$$\delta \mathbf{V} = \delta_s \mathbf{V} = \sum_{k=1}^{k} \delta_s \mathscr{V}_k \tag{6.3}$$

Another case leading to important simplifications is obtained for *quasi-isothermal systems*. In this case it is assumed that the excess temperature  $\theta_k$  of the cells is small compared to the thermal well temperature  $T_r$ .

Consider again the relative virtual dissipation

$$\sum_{k=1}^{k} T_k \delta s_k^* = \sum_{k=1}^{k} (T_r + \theta_k) \delta s_k^*$$
(6.4)

In first approximation we may neglect  $\theta_k$ . Hence as in equation (6.1) we may write

$$\sum_{k=1}^{k} T_k \delta s_k^* = T_r S' \tag{6.5}$$

Again the relative and intrinsic dissipations are the same. Hence the variational principle (5.6) with intrinsic dissipation becomes

$$\delta_{\mathsf{R}}\mathsf{V} + \sum_{i}^{i}\mathsf{I}_{i}\delta q_{i} - \sum_{i}^{i}\mathsf{Q}_{i}\delta q_{i} + \mathsf{T}_{r}\delta\mathsf{S}'$$
(6.6)

where  $\delta_{\mathbf{R}} \mathbf{V}$  is the restricted variation which excludes  $\delta s_k^*$  while

$$\mathbf{T}_{\mathbf{r}}\delta\mathbf{S}' = \sum_{i=1}^{i} \mathbf{X}_{i}'\delta q_{i}$$

is the same virtual relative dissipation as in equation (3.6).

This result is essentially equivalent to that obtained in 1954-55 by the author [1-2] in the more restricted linear context.

#### 7. QUASI-REVERSIBLE SYSTEMS

MINIMUM ENTROPY PRODUCTION AND MINIMUM DISSIPATION

Consider the case where the evolution of a system is "almost reversible". As we have seen any instantaneous configuration may be considered as a virtual equilibrium state under reversed dissipative forces  $-X'_i$  and the reversed inertia forces  $-I_i$ . If we now apply the forces  $X'_i$ , the equilibrium is disturbed and associated velocities  $\dot{q}_i$ 

are produced. For an almost reversible systems, the departure from virtual equilibrium will be small, and *Onsager's principle will generally be applicable*. As already shown [1-3] this may be expressed in the form

$$\mathbf{X}'_i = \mathbf{R}'_i = \frac{\partial \mathbf{D}'}{\partial \dot{q}_i} \tag{7.1}$$

where

$$D' = \frac{1}{2} \sum_{ij}^{ij} b_{ij}(q_k) \dot{q}_i \dot{q}_j$$
(7.2)

is a dissipation function with coefficients  $b_{ij}$  depending on the instantaneous state  $q_k$ . By definition a system for which this property is valid will be called *quasi-reversible*.

Minimum entropy production. By equation (7.1) and application of Euler's theorem on quadratic forms we obtain

$$S'T_r = \sum_{i=1}^{i} X'_i \dot{q}_i = \sum_{i=1}^{i} \frac{\partial D'}{\partial \dot{q}_i} \dot{q}_i = 2D'$$
(7.3)

Hence 2D' represents what we have called the relative dissipation with reference to a thermal well at the temperature  $T_r$ . It is proportional to the total rate of entropy production  $\dot{S}'$ .

According to basic evolutive equations (3.15) the dissipative forces  $X'_i$  are equal to

$$X'_i = -F'_i - I_i + Q_i$$
 (7.4)

They are thus expressed in terms of the frozen instantaneous state including inertia forces.

Using the fixed frozen values (7.4) of  $X'_i$  equations (7.1) lead to the minimum condition

$$D' = minimum \tag{7.5}$$

where only the velocities are varied while satisfying the constraint

$$\sum_{i=1}^{i} X_i' \dot{q}_i = \text{Const}$$
(7.5)

Since according to (7.3) D' is proportional to S' we conclude that at every instant the *velocity direction* is determined by a condition of *minimum rate of entropy production* provided the velocities satisfy the constraint (7.5). This theorem was already derived earlier (1955) in the context of linear thermodynamics [2]. As can be seen from the foregoing considerations its validity for quasi-reversible non-linear systems is immediate as already pointed out for non-linear heat conduction [5].

Minimum dissipation. The foregoing minimum principle is expressed in terms of entropy production. A similar theorem may be expressed in terms of intrinsic dissipation. We write the collective virtual intrinsic dissipation (4.3) as

$$\sum_{k=1}^{k} \mathbf{T}_{k} \delta s_{k}^{*} = \sum_{l=1}^{kl} X_{l}^{k} \delta q_{l}^{k}$$
(7.6)

where  $q_i^k$  are the state variables of  $k^{th}$  cell. On the other hand the variation  $\delta q_i^k$  are linear functions of the independent variations  $\delta q_i$  of the generalized coordinates

$$\delta q_i^k = \sum_{i=1}^{i} \alpha_{ii}^k \delta q_i \tag{7.7}$$

this implies

$$\dot{q}_{l}^{k} = \sum_{i}^{i} \alpha_{li}^{k} \dot{q}_{i} \tag{7.8}$$

Hence

$$\alpha_{ii}^{k} = \frac{\partial \dot{q}_{i}^{k}}{\partial \dot{q}_{i}} \tag{7.9}$$

In a quasi-reversible system we apply Onsager's principle by writing the dissipative forces in each cell as

$$X_l^k = \frac{\partial \mathscr{D}^k}{\partial \dot{q}_l^k} \tag{7.10}$$

where  $\mathscr{D}_k$  is a quadratic form in the variables  $\dot{q}_l^k$  with coefficients depending on  $q_l^k$ . Substituting the values (7.7) (7.9) and (7.10) into. expression (7.6) we obtain

$$\sum_{k=1}^{k} \mathbf{T}_{k} \delta s_{k}^{*} = \sum_{k=1}^{ilk} \frac{\partial \mathcal{D}^{k}}{\partial \dot{q}_{k}} \frac{\partial \dot{q}_{l}^{k}}{\partial \dot{q}_{i}} \delta q_{i} = \sum_{k=1}^{i} \mathbf{X}_{i} \delta q_{i}$$
(7.11)

where the intrinsic dissipative force is

$$X_i = \frac{\partial D}{\partial \dot{q}_i} \tag{7.12}$$

and D is an intrinsic dissipation function given by

$$\mathbf{D} = \sum_{k=1}^{k} \mathscr{D}^{k} = \frac{1}{2} \sum_{j=1}^{ij} b(q_{k}) \dot{q}_{i} \dot{q}_{j}$$
(7.13)

It should be noted that (7.6) along with (7.12) and application of Euler's theorem leads to

$$2\mathbf{D} = \sum_{k=1}^{k} \mathbf{T}_{k} \dot{\mathbf{s}}_{k}^{*} \tag{7.14}$$

We now go back to equations (5.10) of the system. The frozen values of  $X_i$  may be expressed as

$$R_i = X_i = -F_i - I_i + Q_i$$
(7.15)

A consequence of equation (7.12) is the minimum property

$$D = minimum \tag{7.16}$$

where only the velocities  $\dot{q}_i$  are varied while satisfying the constraint

$$\sum_{i=1}^{i} X_i \dot{q}_i = \text{Const.}$$
(7.17)

Hence for a given instantaneous configuration the velocity direction is determined by a condition of *minimum intrinsic dissipation* provided the velocities satisfy the constraint (7.17).

For an isothermal or quasi-isothermal system, relative and intrinsic dissipations coincide, and minimum intrinsic dissipation implies minimum rate of entropy production.

#### 8. LAGRANGIAN EQUATIONS

As in classical mechanics the variational principles lead to Lagrangian equations with generalized coordinates. This has been developed and applied extensively in earlier work mainly in the linear context [1-3, 6] and in some special cases of non-linearity [5-6]. As in mechanics we may distinguish between holonomic and non-holonomic systems.

Holonomic systems. We shall call a system thermodynamically holonomic if the system may be defined by independent state variables  $q_i$  in such a way that all mechanical as well as mass energy and other conservation constraints are obeyed. In practice there are numerous cases where this is applicable either exactly or as a first approximation.

For instance this will generally be the case for isothermal and quasiisothermal systems. Also it will be approximately valid for weakly irreversible systems where the produced entropy adds only a small contribution to the state variables.

We shall apply the principle of virtual intrinsic dissipation (5.6). The restricted variations of the collective potential is

$$\delta_{\mathbf{R}}\mathbf{V} = \sum_{i=1}^{i} \frac{\partial \mathbf{V}}{\partial q_{i}} \delta q_{i}$$
(8.1)

If the body forces are derived from a potential G we may replace  $Q_i \delta q_i$  by

$$\left(\mathbf{Q}_i - \frac{\partial \mathbf{G}}{\partial q_i}\right) \delta q_i \tag{8.2}$$

Also by a well known classical procedure we obtain using mass displacements ( $\mathcal{T}$  = kinetic energy)

$$\sum_{i=1}^{i} \mathbf{I}_{i} \delta q_{i} = \left[ \frac{d}{dt} \left( \frac{\partial \mathcal{F}}{\partial \dot{q}_{i}} \right) - \frac{\partial \mathcal{F}}{\partial q_{i}} \right] \delta q_{i}$$
(8.3)

Introducing these results in the variational principle (5.6) the evolutive equations are written in the form

$$\frac{d}{dt} \left( \frac{\partial \mathscr{F}}{\partial \dot{q}_i} \right) - \frac{\partial \mathscr{F}}{\partial q_i} + \mathbf{R}_i + \frac{\partial \mathscr{P}}{\partial q_i} = \mathbf{Q}_i \tag{8.4}$$

where

$$\mathscr{P} = \mathbf{V} + \mathbf{G} \tag{8.5}$$

is a mixed collective potential. If the system is quasi-reversible with an intrinsic dissipation function D given by (7.13) we put

$$\mathbf{R}_i = \frac{\partial \mathbf{D}}{\partial \dot{q}_i} \tag{8.6}$$

Equations (8.4) are the Lagrangian equations for the generalized coordinates  $q_i$ . The importance of these equations is due to the fact that a large variety of thermodynamic systems obey such equations either rigorously or approximately.

Non-holonomic systems. For systems which are not holonomic Lagrangian equations of a special type may be derived by using Lagrangian multipliers as in classical mechanics. Consider the case when the entropy produced in the cells  $s_k^*$  is not negligible and must be considered as additional state variables required for the description of the system.

The distribution of  $s_k^*$  among the cells constitutes a scalar field which is related to  $q_i$  and  $\dot{q}_i$  by equations of the type

$$\dot{s}_k^* = \dot{s}_k^*(q_i, \dot{q}_i, \mathscr{S}_k) \tag{8.7}$$

These relations between  $s_k^*$  and  $q_i$  are not holonomic i.e. they are essentially a differential relation which must be added to the Lagrangian equations (8.4).

Linear thermodynamics. Over the last twenty years, the Lagrangian equations for this case have been developed and applied extensively with particular reference to viscoelasticity [1-3, 6]. The initial equilibrium state is represented by  $q_i = \mathcal{P} = \mathcal{T} = D = Q_i = 0$ . The generalized coordinates  $q_i$  represent "small" deviations from equilibrium. The system is holonomic since  $s^*$  is of second order and the Lagrangian equations become

$$\frac{d}{dt}\left(\frac{\partial\mathcal{F}}{\partial\dot{q}_i}\right) + \frac{\partial\mathbf{D}}{\partial\dot{q}_i} + \frac{\partial\mathcal{P}}{\partial q_i} = \mathbf{Q}_i \tag{8.8}$$

where

$$\mathcal{T} = \frac{1}{2} \sum_{ij}^{ij} m_{ij} \dot{q}_i \dot{q}_j$$
$$D = \frac{1}{2} \sum_{ij}^{ij} b_{ij} \dot{q}_i \dot{q}_j$$
$$\mathcal{P} = \frac{1}{2} \sum_{ij}^{ij} a_{ij} q_i q_j$$
(8.9)

with constant coefficients  $m_{ij}b_{ij}a_{ij}$ . The quadratic forms  $\mathcal{T}$  and D are positive definite. The same equations are applicable to thermoelasticity and heat conduction as developed earlier [3, 5].

Internal coordinates. In many cases the *n* coordinates  $q_i$  may be grouped in two categories. In one group the coordinates  $q_i(i = 1, 2, ..., k)$  are external or observed, in the other (i = k + 1...n), constituting what we have called internal coordinates, [1] they are not directly observed. The internal coordinates are those for which  $\mathcal{T} = 0$  and  $Q_i = 0$ .

Hence they obey the equations

$$\frac{\partial \mathbf{D}}{\partial q_i} + \frac{\partial \mathscr{P}}{\partial q_i} = 0 \quad (i = k + 1, \dots n) \tag{8.10}$$

It was shown [1-3, 5] that in this case the driving forces corresponding to the external coordinates  $q_i$  are expressed by

$$Q_i = \sum_{j=1}^{j} Z_{ij} q_j \ (i, j = 1, 2, \dots k)$$
 (8.11)

with

$$Z_{ij} = Z_{ji} = \sum_{ij}^{s} D_{ij}^{(s)} \frac{p}{p+r_s} + D_{ij} + p D_{ij}' + p^2 m_{ij}$$
(8.12)

where  $r_{i} > 0$  and  $D_{ij}^{(s)}D_{ij}D_{ij}'$  and  $m_{ij}$  are all positive definite. The symbol p denotes the time derivative operator d/dt. For harmonic time dependence it becomes  $p = i\omega$  ( $\omega$  = circular frequency). Equations (8.11) are integro-differential and represent a system with *heredity. Viscoelastic solids* are a particular case [1][4].

Non-oscillatory character of unstable equilibrium. The initial state  $(q_i = 0)$  may be an unstable equilibrium. This will be the case if  $\mathcal{P}$  may assume negative values. Assuming no driving forces  $(Q_i = 0)$  it was shown that the instability is non-oscillatory [6-7] (p real positive). It should be noted that the instability due to

$$\mathscr{P} < 0 \tag{8.13}$$

may be the result of either V < 0 or G < 0 or both, i.e. it may be purely thermodynamic, purely mechanical, or of *mixed* mechanical and thermodynamic character.

Unstable equilibrium at minimum entropy. Consider the particular case where inertia effects are negligible for either small acceleration of masse, while G = 0. In this case instability implies

$$\mathscr{P} = \mathbf{V} < \mathbf{0} \tag{8.14}$$

On the other hand for this case  $W_t = 0$  and equation (2.11) implies

$$\mathbf{V} = -\mathbf{S}'\mathbf{T}_{\mathbf{r}} \tag{8.15}$$

In other words a maximum value of V at equilibrium implies a minimum value of the total entropy S'. This type of instability was discussed in detail earlier [7, 4]. In this form the instability may be interpreted in terms of probabilistic statistical concepts.

### 9. NON-LINEAR THERMOELASTICITY AND HEAT CONDUCTION AS ILLUSTRATIONS

Quasi-isothermal non-linear thermoelasticity was treated earlier [4]. It is extended hereafter to the completely general case. Consider an elastic continuum. A material point initially of coordinates  $x_i$  is displaced and acquires the coordinates

$$\xi_i = \xi_i(x_k, t) = x_i + u_i(x_k, t)$$
(9.1)

The finite deformation is described by Green's tensor (with dummy indices and the summation convention)

$$\gamma_{ij} = \frac{1}{2}(a_{ij} + a_{ji} + a_{ki}a_{kj}) \tag{9.2}$$

with

$$a_{ij} = \frac{\partial u_i}{\partial x_j} \tag{9.3}$$

The cells are now infinitesimal elements and summations are replaced by volume integrals.

The collective potential may be expressed as

$$\mathbf{V} = \int_{\Omega} \mathscr{V}(\gamma_{ij}, \mathscr{S}, x_i) d\Omega \tag{9.4}$$

where  $d\Omega = dx_1 dx_2 dx_3$  and  $\Omega$  is the initial volume. The term *thermo*elastic potential [4-5, 7] has also been used for V in the present case. The cell potential  $\mathscr{V}$  and its entropy  $\mathscr{S}$  is for a unit initial volume at the location  $x_i$ .

The cell entropy may be split into a supplied entropy s and a produced entropy  $s^*$ . We write

$$\mathscr{S} = s + s^* \tag{9.5}$$

When entropy production is due to thermal conduction, the behavior is quasi-reversible. Onsager's principle is applicable leading to a dissipation function represented by a quadratic function of the rate variables. Results are expressed as follows [4].

We have introduced [3] [4] an entropy displacement field  $S_i$  defined by the equation

$$\dot{S}_i = \frac{\dot{H}_i}{T}$$
(9.6)

where T is the temperature at the point  $\xi_i$  while  $H_i$  denotes the rate of heat flow across a material surface which before deformation is of unit area and normal to  $x_i$ . It was shown that, with the summation convention

$$s = -\frac{\partial \mathbf{S}_i}{\partial x_i} \tag{9.7}$$

and the rate of entropy production per unit initial volume is

$$\dot{s}^* = \lambda_{ij} \dot{S}_i \dot{S}_j \tag{9.8}$$

where

$$\lambda_{ij} = \lambda_{ij} = \lambda_{ij}(\gamma_{lk}, \mathscr{G}\chi_i) \tag{9.9}$$

is the inverse of the thermal conductivity tensor referred to a deformed element and defined by

$$\frac{\partial \mathbf{T}}{\partial x_i} = -\lambda_{ij} \dot{\mathbf{H}}_j \tag{9.10}$$

The virtual intrinsic dissipation is

$$\int_{\Omega} T\delta s^* d\Omega = \int_{\Omega} T\lambda_{ij} \dot{S}_j \delta S_i d\Omega$$
(9.11)

Field equations. The thermoelastic analysis in the present formulation involves the evaluation of the displacement field  $u_i$  and the entropy field  $\mathscr{S}$ . Field equations are readily obtained by applying the variational principle (5.5) which involves the intrinsic dissipation (9-11) and the restricted variation  $\delta_{\mathbf{R}} \mathbf{V}$ . In the present case it is written

$$\int_{\Omega} (\delta_{\mathbf{R}} \mathscr{V} + \rho \overset{\prime \prime}{u}_{i} \delta u_{i} - \rho \mathscr{B}_{i} \delta u_{i} + \mathrm{T} \lambda_{ij} \dot{\mathrm{S}}_{j} \delta \mathrm{S}_{i}) d\Omega \qquad (9.12)$$

where  $\rho$  is the initial mass density at point  $x_i$  and  $\mathscr{B}_i$  is the body force per unit mass at the displaced point  $x_i + u_i$ . Since  $\delta_R \mathscr{V}$  excludes the variation due to  $\delta s^*$  we write

- 24 -

$$\delta_{\mathbf{R}}\mathbf{V} = \int_{\Omega} \delta_{\mathbf{R}} \mathscr{V} d\Omega = \int_{\Omega} \left( \frac{\partial \mathscr{V}}{\partial a_{ij}} \delta a_{ij} + \frac{\partial \mathscr{V}}{\partial s} \delta s \right) d\Omega \tag{9.13}$$

We integrate by parts after introducing the values (9.3) and (9.7) for  $a_{ij}$  and s. These relations play the role of holonomic constraints expressing continuity and conservation. This yields

$$\delta_{\mathbf{R}}\mathbf{V} = \int_{\Omega} \delta_{\mathbf{R}} \mathscr{V} d\Omega = \int_{\Omega} \left( -\frac{\partial}{\partial x_j} \left( \frac{\partial \mathscr{V}}{\partial a_{ij}} \right) \delta u_i + \frac{\partial}{\partial x_i} \left( \frac{\partial \mathscr{V}}{\partial s} \right) \delta \mathbf{S}_i \right) d\Omega \qquad (9.14)$$

Variations  $\delta u_i$  and  $\delta S_i$  are arbitrary and assumed to be applied only inside  $\Omega$ .

Hence substitution of the value (9.14) into the variational principle (9.12) provides the six field equations

$$\frac{\partial}{\partial x_j} \left( \frac{\partial \mathscr{V}}{\partial a_{ij}} \right) + \rho \mathscr{B}_i = \rho \overset{\prime\prime}{u}_i \tag{9.15}$$

$$\frac{\partial}{\partial x_i} \left( \frac{\partial \mathscr{V}}{\partial s} \right) + \mathcal{T} \lambda_{ij} \dot{\mathbf{S}}_j = 0 \tag{9.16}$$

Since  $\partial \mathscr{V}/\partial s = \theta$  equations (9.16) are the heat conduction equations.

Both T and  $\lambda_{ij}$  are functions of  $\gamma_{ij}, x_i$  and  $\mathscr{S}$ . Hence the six equations (9.15) and (9.16) contain seven unknowns,  $u_i$ ,  $S_i$  and  $s^*$ . A seventh equation is given by expression (9.8) for the local rate of entropy production  $\dot{s}^*$ , in terms of  $S_{ij}, s^*, x_i$  and  $\gamma_{ij}$ .

Lagrangian equations and holonomic approximation. If the contribution of the produced entropy  $s^*$  to the state variable  $\mathscr{S}$  remains small in comparison with the contribution of the supplied entropy s, the cell entropy  $\mathscr{S}$  may be written

$$\mathcal{S} = s = -\frac{\partial \mathbf{S}_i}{\partial x_i} \tag{9.17}$$

and the system becomes holonomic. It is then completely described by two fields of material displacements  $u_i$  and entropy displacements  $S_i$ . They are written

$$u_{i} = u_{i}(q_{1}q_{2} \dots q_{n}x_{k})$$
  

$$S_{i} = S_{i}(q_{1}q_{2} \dots q_{n}x_{k})$$
(9.18)

in terms of generalized coordinates  $q_i$  which are unknown functions of the time to be determined by Lagrangian equations (8.4). Note that in expressions (9.18) the time could also be included as a variable M. A. Biot

in addition to  $q_i$  and  $x_i$ . In order to obtain the Lagrangian formulation we need to evaluate the collective or thermoelastic potential

$$\mathbf{V} = \int_{\Omega} \mathscr{V}(\gamma_{ij}, \mathbf{s}, \mathbf{x}_i) d\Omega = \mathbf{V}(q_1 q_2 \cdots q_n)$$
(9.19)

We also need the kinetic energy  $\mathcal{T}$  and the intrinsic dissipation function D. They are written

$$\mathscr{T} = \frac{1}{2} \int_{\Omega} \rho \dot{u}_i \dot{u}_i d\Omega = \frac{1}{2} m_{ij} (q_1 q_2 \cdots q_n) \dot{q}_i \dot{q}_j \qquad (9.20)$$

$$\mathbf{D} = \frac{1}{2} \int_{\Omega} \mathbf{T} \lambda_{ij} \dot{\mathbf{S}}_i \dot{\mathbf{S}}_j d\Omega = \frac{1}{2} b_{ij} (q_1 q_2 \cdots q_n) \dot{q}_i \dot{q}_j \qquad (9.21)$$

where the temperature T is expressed as a function of  $\gamma_{ij}$ , s and  $x_k$ 

$$T = T(\gamma_{ij}, s, x_k) \tag{9.22}$$

It should be pointed out that in the present case the driving forces  $Q_i$  at the boundary may be expressed as *mixed thermo-mechanical forces*. This can be seen by considering the boundary to be in contact with thermal reservoir's at various temperatures. The cell potential of these reservoir's may be considered as contributing additional terms to the collective potential V of the system. This amounts to replacing  $\partial V/\partial q_i$  by  $(n_k = unit normal)$ 

$$\frac{\partial \mathbf{V}}{\partial q_i} + \int_{\mathbf{A}} \theta n_k \frac{\partial \mathbf{S}_k}{\partial q_i} d\Omega \tag{9.23}$$

The surface integral at the boundary A is the contribution of the driving thermal reservoir's to the potential term. However it may be considered as part of the driving forces by transferring it to the right hand side of the Lagrangian equations (5.4) putting

$$Q_{i} = -\int_{A} \theta n_{k} \frac{\partial S_{k}}{\partial q_{i}} dA + \int_{A} f_{k} \frac{\partial u_{k}}{\partial q_{i}} dA.$$
(9.24)

The first integral represents the thermal driving forces, and the second the purely mechanical part due to driving forces  $f_i$  per unit initial area at the boundary. Note that  $\theta$  may be time dependent temperatures, since the driving reservoirs are fictitious and defined in terms of instantaneous values.

Finally the body forces may be derived from a potential G such as a gravity force potential. This amount the replacing V by the mixed collective potential  $\mathscr{P} = V + G$  already considered (8.5) in the general case.

With these results the Lagrangian equations of the thermoelastic medium are written

$$\frac{d}{dt} \left( \frac{\partial \mathscr{F}}{\partial \dot{q}_i} \right) - \frac{\partial \mathscr{F}}{\partial q_i} + \frac{\partial \mathbf{D}}{\partial \dot{q}_i} + \frac{\partial \mathscr{P}}{\partial q_i} = \mathbf{Q}_i \tag{9.25}$$

Note that if the holonomic approximation is not sufficient, we may use an iteration procedure by which the holonomic approximation is considered as a first step. The values of  $q_i$  thus obtained are used to determine  $\dot{s}^*$  from equations (9.8) and (9.18). A simple time quadrature yields  $s^*$  as a function of time. We then replace s by  $\mathscr{S} = s + s^*$ in the Lagrangian equations and derive new values of  $q_i$  as a second approximation. The procedure may be repeated.

Quasi isothermal case and elastic instability. If the temperature increments  $\theta$  are small, we may write

$$S_i = \frac{H_i}{T_r}$$
(9.26)

where  $H_i$  represents a heat displacement. The dissipation function becomes

$$\mathbf{D} = \frac{1}{2\mathrm{T}_r} \int_{\Omega} \lambda_{ij} \mathbf{H}_i \mathbf{H}_j d\Omega$$
(9.27)

and coincides with the relative dissipation function (7.2). This case was analyzed in detail earlier [4] and applied to a discussion of the nature of thermoelastic instability for the linearized case of an initially stressed elastic medium as already mentioned above in connection with equations (8.13) and (8.15).

Non-linear heat conduction. The case of pure heat conduction is obtained by assuming zero deformation  $(\gamma_{ij} = 0)$  and zero kinetic

energy ( $\mathcal{F} = 0$ ). This case was treated and applied extensively in a sequence of papers originating in 1957 and collected in a monograph [5]. The principle of virtual dissipation may be simplified in this case leading to a rigorously holonomic system and corresponding Lagrangian equations.

We apply the variational principle (9.12) with restricted variations and intrinsic dissipation.

It becomes

$$\int_{\Omega} (\delta_{\mathbf{R}} \mathscr{V} + \mathrm{T} \lambda_{ij} \dot{\mathrm{S}}_j \delta \mathrm{S}_i) d\Omega$$
(9.28)

where

$$\delta_{\mathbf{R}}\mathscr{V} = \theta \delta s = -\theta \frac{\partial}{\partial x_i} \delta \mathbf{S}_i = -\theta \frac{\partial}{\partial x_i} \left( \frac{\delta \mathbf{H}_i}{\mathbf{T}} \right)$$
(9.29)

We assume arbitrary variations  $\delta H_i$  to be different from zero inside  $\Omega$ . We introduce the values (9.29) into the variational principle (9.28) and integrate by parts. This yields

$$\int_{\Omega} \left( \frac{\partial \theta}{\partial x_i} + \lambda_{ij} \dot{\mathbf{H}}_j \right) \frac{\delta \mathbf{H}_i}{\mathbf{T}} d\Omega = 0$$
(9.30)

Since the variations  $\delta H_i$ , hence also  $\delta H_i/T$  are arbitrary, this results is equivalent to

$$\int_{\Omega} \left( \frac{\partial \theta}{\partial x_i} \delta \mathbf{H}_i + \lambda_{ij} \dot{\mathbf{H}}_j \delta \mathbf{H}_i \right) d\Omega = 0$$
(9.31)

Finally another integration by parts yields

$$\int_{\Omega} (\theta \delta h + \lambda_{ij} \dot{\mathbf{H}}_j \delta \mathbf{H}_i) d\Omega = 0$$
(9.31)

with

$$h = -\frac{\partial \mathbf{H}_i}{\partial x_i} \tag{9.32}$$

This quantity was referred to as the heat content [5]. In the present case of pure heat conduction it plays the role of a state variable related to the temperature.

We write

$$\theta = \theta(h) \tag{9.33}$$

By adding thermal reservoirs at the boundary the term  $\theta \delta h$  may be considered to include these reservoirs. It amounts to adding a surface integral at the boundary, and equation (9.31) becomes

$$\int_{\Omega} (\theta \delta h + \lambda_{ij} \dot{H}_j \delta H_i) d\Omega = -\int_{\Lambda} \theta n_k \delta H_k dA \qquad (9.34)$$

This is the same variational principle as proposed earlier by the author for thermal conduction in 1957 (see [5]).

It leads to Lagrangian equations developed and discussed extensively in a book [5]. The heat displacement field is expressed in terms of generalized coordinates

$$\mathbf{H} = \mathbf{H}_i(q_1 q_2 \dots q_n x_k) \tag{9.35}$$

The Lagrangian equations for  $q_i$  derived from the variational principle (9.34) are

$$\frac{\partial \mathbf{V}}{\partial q_i} + \frac{\partial \mathbf{D}}{\partial \dot{q}_i} = \mathbf{Q}_i \tag{9.36}$$

where

$$\mathbf{V} = \int_{\Omega} d\Omega \int_{0}^{\theta} \theta dh \tag{9.37}$$

was called the thermal potential while

$$\mathbf{D} = \frac{1}{2} \int_{\Omega} \lambda_{ij} \dot{\mathbf{H}}_i \dot{\mathbf{H}}_j d\Omega \tag{9.38}$$

is a thermal dissipation function and

$$\mathbf{Q}_{i} = -\int_{\Lambda} \theta n_{k} \frac{\partial \mathbf{H}_{k}}{\partial q_{i}} d\mathbf{A}$$
(9.39)

is the generalized thermal force.

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