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ON A UNIFIED THERMODYNAMIC APPROACH TO A LARGE CLASS OF INSTABILITIES OF DISSIPATIVE CONTINUA

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As a contribution toward unification and interdisciplinary perspective it seems appropriate here to call attention to an extensive body of results established by another school regarding the thermodynamic aspects of instability *leading to dissipative structures*. The theory deals with the linear irreversible thermodynamics of deviations from a state of equilibrium which is unstable, as distinguished from unstable perturbations in the vicinity of a nonequilibrium steady state. However, as pointed out below, special cases of instability of steady-state flows may also be analyzed by similar methods.

This general approach provides a systematic thermodynamic analysis of continua which are initially stressed in a state of unstable equilibrium. Typical classical examples are elastic buckling in compression, and a viscous fluid in equilibrium in a gravity field with an unstable density gradient. In general perturbations of the unstable equilibrium obey the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right)+\frac{\partial \mathcal{D}}{\partial \dot{q}_i}+\frac{\partial \mathcal{P}}{\partial q_i}=0 \tag{1}$$

where $\mathcal{T}$ is the kinetic energy, $\mathcal{D}$ the dissipation function defined in terms of entropy production, and $\mathcal{P}$ is a new mixed mechanical and thermodynamic potential different from the classical thermodynamic potentials. The generalized coordinates q_i define the nonisothermal perturbations from the initial state of stressed equilibrium. Instability arises from the fact that $\mathcal{P}$ is not positive-definite.

The theory was developed over the last 20 years starting in 1954[1–3] with a Lagrangian-variational formulation of irreversible thermodynamics, followed by a sequence of applications and further developments, some of which are given in the reference section[4–17]. The theory of viscoelasticity of initially stressed solids and fluids including thermodynamic foundations was presented in systematic fashion in a book puplished in 1965.[18]

With the use of internal coordinates the thermodynamics embodied in (1) are readily applicable not only to viscous fluids but to viscoelastic solids with memory. This aspect of viscoelasticity based on internal coordinates as developed earlier,[1 to 3] was also presented in a book by Fung.[25]

Unstable solutions are proportional to $\exp(pt)$ where according to a basic theorem [Ref. 18, p. 441] p is real and positive. Thus for systems governed by (1) the incipient instability is *nonoscillatory*.

The theorem is applicable to viscoelastic solids if we assume that the hereditary behavior is due to a large number of unobserved internal coordinates included in the generalized coordinates q_i governed by (1). Note that the theorem assumes an initial state of equilibrium but an unstable one. It should be noted that it is still applicable to instability in the vicinity of a state of steady flow in cases were the system may be approximately represented by (1) as occurs in many problems of viscous buckling of solids. However other types of problems with steady heat flow may lead to oscillatory instability.[24]

It is not possible here to outline all the numerous applications and therefore only a few of the highlights will be mentioned. Stability in the thermodynamic context was discussed at a IUTAM colloquium in Madrid in 1955[3] followed by an application to the folding instability of an embedded viscoelastic layer in compression.[4] This brought out the important qualitative role played by thermodynamic principles and the appearance of dissipative structures embodied in the concept of *dominant wavelength*, thereby showing that such structures do not necessarily require a nonlinear thermodynamic behavior. The folding instability of a porous layer and its thermoelastic analogy were also analyzed[15] as an application of the general stability theory of porous dissipative solids.[13]

The instability of a multilayered viscous fluid in a steady state of compressive flow with finite strain subject at the same time to gravity forces was given a general and systematic treatment.[16,18,19] This theory considers small displacement perturbations superposed upon finite displacements which themselves are time dependent and constitute the initial unperturbed steady flow. A general theory of instability was also developed for a multilayered system including materials with couple-stresses and was applied to thinly laminated layers.[22]

Numerous applications have been made to problems of geological folding of layered rock under tectonic stresses and the results have been verified experimentally.[11,12] Such geological features provide another example of dissipative structures. A particularly interesting result is represented by a numerical evaluation of the actual time history of folding of a layered viscous medium in compression starting with a local bell-shaped deviation from a perfectly flat layer.[12,18] The gradual appearance of a dissipative structure in the form of folds may be followed as time goes on, and the wavelength of these folds turns out to be insensitive to the type of initial disturbance. It is also pointed out that a multiplicity of initial disturbances each generating its own structure similar to a sinusoidal wave packet may give rise to mutual interference patterns.

In the special case of destabilizing gravity forces the theory was applied to the formation of salt domes in geophysics, under transient conditions of gradual sedimentation and time dependence of thickness and compaction of the material overlying the salt layer.[20,21] A particularly interesting feature of the results is some kind of degeneracy exhibited by the cell pattern showing that triangular, hexagonal, circular, and many other patterns are equally unstable. Thus the appearance of any particular pattern should be very sensitive to boundary constraints, perturbations of these constraints, and initial conditions. It was shown that a system of localized irregular initial perturbations may produce a system of ring-shaped cells with a mutual interference pattern. Results are in good agreement with observed geophysical structures and geological time scales.

Finally it should be noted that the theory is not restricted to linear perturbations as exemplified by more recent work on nonlinear thermoelasticity, the thermodynamics of elastic instability, and postbuckling behavior.[23]

The Lagrangian approach outlined here provides not only deeper and unified physical insights but also powerful methods of approximate analysis formulated directly in terms of generalized coordinates of complex systems. This procedure does not require any preliminary knowledge of the differential field equations of the system. This is in contrast with the purely formal so-called projection methods based on abstract functional space theories.

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