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FUNDAMENTALS OF GENERALIZED RIGIDITY MATRICES FOR MULTI-LAYERED MEDIA

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ABSTRACT

Rigidity matrices for multi-layered media are derived for isotropic and orthotropic layers by a simple direct procedure which brings to light their fundamental mathematical structure. The method was introduced many years ago by the author in the more general context of dynamics and stability of multi-layers under initial stress. Other earlier results are also briefly recalled such as the derivation of three-dimensional solutions from plane strain modes, the effect of initial stresses, gravity, and couple stresses for thinly laminated layers. The extension of the same mathematical structure and symmetry to viscoelastic media is valid as a consequence of fundamental principles in linear irreversible thermodynamics.

Introduction

In a recent paper (Kausel and Roësset, 1981), stiffness matrices were derived for isotropic multi-layered media starting from the Thomson-Haskell transfer matrices. However, such stiffness matrices were already derived many years ago (Biot, 1963, 1965) and discussed in great detail in the more general context of orthotropic multi-layered solids under initial stress. The derivation was not based on the Thomson-Haskell matrices but used instead a direct approach which is extremely simple and takes advantage of the physical symmetry of the layers. The fundamental mathematical structure of the 4×4 stiffness matrix is thus brought to light, forming a symmetric matrix of six independent terms. The structure is valid for a wide range of physical systems and embodies the basic reciprocity properties of cross impedances of linear conservative systems.

In the present paper, the method has been used to derive the stiffness matrices directly in the particular case of isotropic and orthotropic layers without initial stress, and it is shown how these results are also obtained from the earlier ones. The basic equations for multi-layers are obtained in the form of recurrence equations for displacements at three successive interfaces, and it is recalled how they may be written in compact form as a variational principle including the effect of initial stresses, gravity forces and couple stresses.

Using the earlier developments (Biot, 1966, 1972b, 1974), it is shown how simple plane strain solutions lead to a large class of three-dimensional solutions for transverse isotropy without any additional evaluation of matrix elements.

Finally, for viscoelastic materials whose heredity properties are based on linear thermodynamics with Onsager's reciprocity relations and internal coordinates, it is recalled that all results remain formally valid with complex functions of the frequency replacing the elastic coefficients. The complex matrices exhibit the same mathematical structure as in the elastic case.

Stiffness Matrix of Symmetric and Antisymmetric Modes

A single isotropic elastic layer is analyzed for plane strain in the $x\,y$ plane normal to the layer. The thickness of the layer is h, and the x axis is parallel to the layer and equidistant from the faces. These faces are thus represented by the planes

$y = \pm h/2$. We denote by u and v the displacements along x and y defining the plane strain components as

$$e_{xx} = \frac{\partial u}{\partial x} \quad e_{yy} = \frac{\partial v}{\partial y} \quad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{1}$$

The corresponding stresses are

$$\begin{aligned} \sigma_{xx} &= He_{xx} + (H - 2L)e_{yy} \\ \sigma_{yy} &= (H - 2L)e_{xx} + He_{yy} \\ \sigma_{xy} &= 2Le_{xy} \end{aligned} \tag{2}$$

where $H = \lambda + 2\mu$, $L = \mu$, and λ, μ denote the Lamé constants. We use L instead of μ in order to avoid a change of notation in the more general cases. For harmonic time dependence, solutions are proportional to the factor $\exp(i\omega t)$. This factor may be omitted in all formulas. The dynamic equilibrium equations are then

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho\omega^2 u = 0 \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho\omega^2 v = 0 \end{aligned} \tag{3}$$

where ρ is the density. For the particular case of an isotropic medium, equations (1) to (3) are solved by the classical procedure of decoupling of dilatational and rotational waves. The values of u and v are then obtained by putting

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \tag{4}$$

where the scalars φ and ψ satisfy the two wave equations

$$\begin{gathered} H\nabla^2{}_{\varphi} + \rho\omega^2\varphi = 0 \qquad L\nabla^2{}_{\psi} + \rho\omega^2\psi = 0 \\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \end{gathered} \tag{5}$$

The following solution is immediately derived

$$u = U(ly)\sin lx \qquad v = V(ly)\cos lx \tag{6}$$

where

$$\begin{aligned} U(ly) &= C_1 \sinh \beta_1 ly + C_2 \sinh \beta_2 ly \\ V(ly) &= -C_1\beta_1 \cosh \beta, ly - \frac{C_2}{\beta_2}\cosh \beta_2 ly \end{aligned} \tag{7}$$

with

$$\beta_1{}^2 = 1 - \frac{\omega^2\rho}{Hl^2} \qquad \beta_2{}^2 = 1 - \frac{\omega^2\rho}{Ll^2} \tag{8}$$

and arbitrary constants C_1 and C_2. The solution represents an antisymmetric standing wave of the layer corresponding to flexural deformation of wavenumber l along x such that

$$U\left(\frac{lh}{2}\right) = -U\left(-\frac{lh}{2}\right) = U_a \qquad V\left(\frac{lh}{2}\right) = V\left(-\frac{lh}{2}\right) = V_a. \tag{9}$$

Putting $\gamma = lh/2$ we obtain

$$\begin{aligned} U_a &= C_1 \sinh \beta_1\gamma + C_2 \sinh \beta_2\gamma \\ V_a &= -C_1\beta_1 \cosh \beta_1\gamma - \frac{C_2}{\beta_2}\cosh \beta_2\gamma. \end{aligned} \tag{10}$$

The stresses are obtained by substituting into (2) the values (6) for u and v. They may be written

$$\sigma_{xy} = \tau(ly)\sin lx \qquad \sigma_{yy} = q(ly)\cos lx \tag{11}$$

with the antisymmetric property

$$\tau\left(\frac{lh}{2}\right) = \tau\left(-\frac{lh}{2}\right) = \tau_a \qquad q\left(\frac{lh}{2}\right) = -q\left(-\frac{lh}{2}\right) = q_a. \tag{12}$$

In calculating these values, the result is simplified by using the identity

$$H - L = H\beta_1^2 - L\beta_2^2. \tag{13}$$

We derive

$$\begin{aligned} \frac{\tau_a}{Ll} &= 2C_1\beta_1 \cosh \beta_1\gamma + C_2\left(\beta_2 + \frac{1}{\beta_2}\right)\cosh \beta_2\gamma \\ \frac{q_a}{Ll} &= -C_1(1+\beta_1^2)\sinh \beta_1\gamma - 2C_2 \sinh \beta_2\gamma. \end{aligned} \tag{14}$$

It is a simple matter to eliminate C_1 and C_2 between equations (10) and (14). This yields

$$\begin{aligned} \frac{\tau_a}{Ll} &= a_{11}U_a + a_{12}V_a \\ \frac{q_a}{Ll} &= a_{12}U_a + a_{22}V_a \end{aligned} \tag{15}$$

where

$$\begin{aligned} a_{11} &= \frac{\beta_1^2}{\Delta_a}(1-\beta_2^2) & a_{22} &= \frac{z_1 z_2}{\Delta_a}(1-\beta_2^2) \\ a_{12} &= \frac{1}{\Delta_a}(\beta_1^2 z_2 - \beta_2^2 z_1) - 1 & \Delta_a &= z_1 - \beta_1^2 z_2 \\ z_1 &= \beta_1 \tanh \beta_1 & z_2 &= \beta_2 \tanh \beta_2. \end{aligned} \tag{16}$$

Similarly, we consider the symmetric standing wave such that

$$U\left(\frac{lh}{2}\right) = U\left(-\frac{lh}{2}\right) = U_s \qquad V\left(\frac{lh}{2}\right) = -V\left(-\frac{lh}{2}\right) = V_s$$
$$\tau\left(\frac{lh}{2}\right) = -\tau\left(-\frac{lh}{2}\right) = \tau_s \qquad q\left(\frac{lh}{2}\right) = q\left(-\frac{lh}{2}\right) = q_s. \tag{17}$$

This is a standing wave where the axis of the layer remains fixed while the layer exhibits a change of thickness of wavenumber l along x. We may proceed as for the antisymmetric case. However, we need not repeat the calculation. We note that in *the solution (7), the functions sinh and cosh are interchanged. This is also true for* their derivatives. Hence, the solutions is obtained from the antisymmetric case by simply interchanging the functions sinh and cosh. This amounts to replacing z_1 and z_2 in (16), respectively, by $\beta_1/\tanh \beta_1\gamma = \beta_1^2/z_1$ and $\beta_2/\tanh \beta_2 = \beta_2^2/z_2$. We obtain

$$\frac{\tau_s}{Ll} = b_{11} U_s + b_{12} V_s$$
$$\frac{q_s}{Ll} = b_{12} U_s + b_{22} V_s \tag{18}$$

where

$$b_{11} = \frac{z_1 z_2}{\Delta_s}(1 - \beta_2^2) \qquad b_{22} = \frac{\beta_2^2}{\Delta_s}(1 - \beta_2^2)$$
$$b_{12} = \frac{\beta_2^2}{\Delta_s}(z_1 - z_2) - 1 \qquad \Delta_s = z_2 - z_1\beta_2^2. \tag{19}$$

Note that the 2×2 stiffness matrices of equations (15) and (18) for the antisymmetric and symmetric deformation of the layer are symmetric, i.e., the off-diagonal terms are equal. Hence, for each case, there are only three distinct coefficients a_{ij} and b_{ij}. This symmetry is a fundamental property of mechanical impedances of linear conservative systems. It is a consequence of the existence of a strain energy. In Lagrangian form, the equations of motion are derived from two quadratic forms representing the elastic and kinetic energy which lead to reciprocity properties of impedance coefficients.

It is of interest to derive values of a_{ij} and b_{ij} for the limiting static case of zero frequency ($\omega = 0$). In this case

$$\beta_1 = \beta_2 = 1 \qquad z_1 = z_2. \tag{20}$$

Hence, the matrix elements become indeterminate of the type 0/0. Their true value is obtained by putting

$$\beta_1 = 1 - \epsilon_1 \quad \beta_2 = 1 - \varepsilon_2 \quad \varepsilon_1 = \frac{1}{2}\frac{\omega^2\rho}{Hl^2} \quad \varepsilon_2 = \frac{1}{2}\frac{\omega^2\rho}{Ll^2} \tag{21}$$

where ε_1 and ε_2 are small quantities of the first order. It is a simple matter to expand values to the first order in ε_1 and ε_2. This brings out the common factor $\omega^2\rho/l^2$ which cancels out. By putting

$$\begin{aligned} r &= L/H \\ \Delta_a' &= (1+r)\sinh 2\gamma + 2(1-r)\gamma \\ \Delta_s' &= (1+r)\sinh 2\gamma - 2(1-r)\gamma \end{aligned} \tag{22}$$

we obtain

$$\begin{aligned} a_{11} &= \frac{4}{\Delta_a'}\cosh^2\gamma \quad & a_{22} &= \frac{4}{\Delta_a'}\sinh^2\gamma \quad & a_{12} &= \frac{2}{\Delta_a'}\sinh^2\gamma - 2 \\ b_{11} &= \frac{4}{\Delta_s'}\sinh^2\gamma \quad & b_{22} &= \frac{4}{\Delta_s'}\cosh^2\gamma \quad & b_{12} &= \frac{2}{\Delta_s'}\sinh^2\gamma - 2. \end{aligned} \tag{23}$$

For an incompressible solid, we put $r = 0$ in Δ_a' and Δ_s'. This case is immediately applicable to a viscous incompressible solid in creeping motion with velocities replacing displacements and the viscosity coefficient η replacing L. These results were applied to problems of folding instability of viscous multi-layered solids under compressive stress along the layers (Biot, 1965).

The characteristic equation for flexural oscillations of a single layer is obtained by putting $\tau_a = q_a = 0$ in equation (15). We derive

$$a_{11}a_{22} - a_{12}^2 = 0. \tag{24}$$

After cancelling the common factor $(z_1 - \beta_1^2 z_2)\beta_1$, this becomes

$$4\beta_1\beta_2 \tanh \beta_2\gamma - (1+\beta_2^2)^2 \tanh \beta_1\gamma = 0. \tag{25}$$

For wavelengths, large relative-to-the-thickness $\beta_1\gamma$ and $\beta_2\gamma$ are small, and we may replace the tanh by the first two terms of its power expansion, i.e.,

$$\begin{aligned} \tanh \beta_1\gamma &= \beta_1\gamma - \tfrac{1}{3}\beta_1^3\gamma^3 \\ \tanh \beta_2\gamma &= \beta_2\gamma - \tfrac{1}{3}\beta_2^3\gamma^3. \end{aligned} \tag{26}$$

This yields

$$\tfrac{1}{3}\gamma^2[(1+\beta_2^2)^2\beta_1^2 - 4\beta_2^4] = (1-\beta_2^2)^2. \tag{27}$$

For flexural waves whose phase velocity is small relative to dilatational and shear waves, we may put β_1 and β_2 equal to (21) where ε_1 and ε_2 are small. Keeping only first order terms and cancelling the common factor $\omega^2\rho/l^2$, equation (27) becomes

$$\frac{1}{3}\frac{H-L}{H}Lh^2l^4 = \omega^2\rho. \tag{28}$$

Expressing the elastic coefficients in terms of Young's modulus E and Poisson's ratio ν, (28) is written

$$\frac{1}{12}\frac{E}{1-\nu^2}h^3l^4 = \omega^2\rho h. \tag{29}$$

This result coincides with the equation of flexural oscillations of thin plates (Timoshenko, 1927)

$$\frac{1}{12}\frac{E}{1-\nu^2}h^3\frac{d_v^4}{dx^4} = \omega^2\rho h v \tag{30}$$

where h is the thickness and v the deflection.

The case of thin plates using series expansion of tanh as a limiting case was discussed in detail earlier in the context of stability under initial stress (Biot, 1965). It was shown that the power expansion of tanh must include the first and third order terms.

A similar power expansion of tanh to the third order for the coefficients a_{ij} and b_{ij} for thin layers yields Lagrangian equations of motion with quadratic forms for the potential and kinetic energies.

Complete Stiffness Matrix of a Layer

It was shown (Biot, 1963, 1965) that the stiffness matrix of a layer for standing waves which are neither symmetric nor antisymmetric is obtained immediately from the values of a_{ij} and b_{ij} without any additional algebra. We denote by $U_1 V_1\ \tau_1 q_1$ the values of the displacements and stresses at the top of the layer ($y = h/2$) and by $U_2 V_2 \tau_2 q_2$ the values at the bottom ($y = -h/2$). By superposing symmetric and antisymmetric modes we obtain

$$\begin{aligned} \tau_1 &= \tau_a + \tau_s \\ q_1 &= q_a + q_s. \end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned} \tau_2 &= \tau_a - \tau_s \\ q_2 &= -q_a + q_s. \end{aligned} \tag{32}$$

We may also write

$$\begin{aligned} U_a &= \tfrac{1}{2}(U_1 - U_2) \qquad & U_s &= \tfrac{1}{2}(U_1 + U_2) \\ V_a &= \tfrac{1}{2}(V_1 + V_2) \qquad & V_s &= \tfrac{1}{2}(V_1 - V_2). \end{aligned} \tag{33}$$

We now substitute the values (15) and (18) of $\tau_a q_a \tau_s q_s$ into (31) and (32), and in the result we then substitute the values (33) of $U_a V_a U_s V_s$. We obtain

$$\begin{bmatrix} \tau_1 \\ q_1 \\ -\tau_2 \\ -q_2 \end{bmatrix} = Ll \begin{bmatrix} A_1 & A_2 & -A_4 & A_5 \\ A_2 & A_3 & -A_5 & A_6 \\ -A_4 & -A_5 & A_1 & -A_2 \\ A_5 & A_6 & -A_2 & A_3 \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} A_1 &= \tfrac{1}{2}(a_{11} + b_{11}) \qquad & A_4 &= \tfrac{1}{2}(a_{11} - b_{11}) \\ A_2 &= \tfrac{1}{2}(a_{12} + b_{12}) \qquad & A_5 &= \tfrac{1}{2}(a_{12} - b_{12}) \\ A_3 &= \tfrac{1}{2}(a_{22} + b_{22}) \qquad & A_6 &= \tfrac{1}{2}(a_{22} - b_{22}). \end{aligned} \tag{35}$$

The stiffness matrix of equation (34) reveals its fundamental mathematical structure which was already discovered in earlier work in a more general context (Biot, 1963, 1965). It is symmetric with respect to the diagonal. This is a consequence of the existence of a strain energy. In addition, it contains only six distinct elements. This, in turn, is due to the symmetry of the elastic properties with respect to the middle plane and the separation into symmetric and antisymmetric modes represented by the six elements a_{ij} and b_{ij}. Note that $-\tau_2$ $-q_2$ are given negative signs in order to represent forces acting on the material at the bottom face in the direction of the displacements U_2 and V_2.

As also shown earlier (Biot, 1963, 1965), a compact formulation of equation (34) without using matrices is obtained by introducing the quadratic form

$$\begin{aligned} I = {} & \tfrac{1}{2}A_1(U_1^2 + U_2^2) - A_4U_1U_2 \\ & + \tfrac{1}{2}A_3(V_1^2 + V_2^2) + A_6V_1V_2 \\ & + A_2(U_1V_1 - U_2V_2) + A_5(U_1V_2 - U_2V_1). \end{aligned} \tag{36}$$

Equation (34) is then written

$$\begin{aligned} \tau_1 &= Ll\frac{\partial I}{\partial U_1} & -\tau_2 &= Ll\frac{\partial I}{\partial U_2} \\ q_1 &= Ll\frac{\partial I}{\partial V_1} & -q_2 &= Ll\frac{\partial I}{\partial V_2}. \end{aligned} \tag{37}$$

The particular case of a vanishing wavelength or infinite layer thickness is obtained by putting $\gamma = \infty$ in the values of z_1 and z_2. Consider the case where β_1 and β_2 are real. Without loss of generality, we may choose β_1 and β_2 as positive. Hence, in this case for $\gamma = \infty$

$$\begin{aligned} \tanh \beta_1\gamma &= 1 & \tanh \beta_2\gamma &= 1 \\ z_1 &= \beta_1 & z_2 &= \beta_2. \end{aligned} \tag{38}$$

Substitution of these values into expressions (16) and (19) yields

$$\begin{aligned} a_{11} = b_{11} &= \frac{\beta_1}{1 - \beta_1\beta_2}(1 - \beta_2^2) \\ a_{22} = b_{22} &= \frac{\beta_2}{1 - \beta_1\beta_2}(1 - \beta_2^2) \\ a_{12} = b_{12} &= \frac{\beta_2(\beta_1 - \beta_2)}{1 - \beta_1\beta_2} - 1. \end{aligned} \tag{39}$$

Equation (34) becomes

$$\begin{bmatrix} \tau_1 \\ q_1 \\ \tau_2 \\ q_2 \end{bmatrix} = Ll \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & -a_{11} & a_{12} \\ 0 & 0 & a_{12} & -a_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix}. \tag{40}$$

Hence, the coupling between the two faces of the layer disappears. The upper 2 ×

2 matrix yields the case of the lower half-space, and the lower 2×2 matrix that of the upper half-space. The case where β_1 or β_2 or both are imaginary may be considered as a limiting case by introducing a vanishingly small damping in the matrices. In this case, choosing β_1 and β_2 to be in the first quadrant, equations (39) and (40) remain valid. It is quite simple of course to solve the problem directly for the half-space. Results confirm the values (39) and yield equation (40).

Recurrence equations for a multi-layered system are readily obtained by a general procedure (Biot, 1963, 1965) using the foregoing results. We denote $A_1^i\ A_2^i\ A_3^i\ A_4^i\ A_5^i\ A_6^i$ the elements of the rigidity matrix of the ith layer. Correspondingly, the displacements at the upper face of this layer are denoted by $U_i\ V_i$ and at the bottom face by $U_{i+1}\ V_{i+1}$. The latter are equal to the upper face displacements of the $(i + 1)$th layer immediately below. Using expression (37) for the stresses acting on the layers i and $i + 1$, we equate the stresses acting at their common interface. This yields

$$\begin{aligned} &\frac{\partial}{\partial U_{i+1}}(L_i I_i + L_{i+1} I_{i+1}) = 0 \\ &\frac{\partial}{\partial V_{i+1}}(L_i I_i + L_{i+1} I_{i+1}) = 0 \end{aligned} \tag{41}$$

with the quadratic form

$$\begin{aligned} I_i = {} & \tfrac{1}{2} A_1^i(U_i^2 + U_{i+1}^2) - A_4^i U_i U_{i+1} \\ & + \tfrac{1}{2} A_3^i(V_i^2 + V_{i+1}^2) + A_6^i V_i V_{i+1} \\ & + A_2^i(U_i V_i - U_{i+1} V_{i+1}) + A_5^i(U_i V_{i+1} - U_{i+1} V_i). \end{aligned} \tag{42}$$

These are recurrence equations for the two displacements at three successive interfaces. Equations for end conditions at top and bottom of the multi-layered system, whether free or coupled to a half-space, are easily added. They have been discussed earlier in a different context (Biot, 1963, 1965).

Further formal simplification of the recurrence equation is obtained by introducing the quadratic form

$$\mathscr{I} = \sum L_i I_i \tag{43}$$

Equation (41) then becomes

$$\frac{\partial \mathscr{I}}{\partial U_i} = 0 \qquad \frac{\partial \mathscr{I}}{\partial V_i} = 0. \tag{44}$$

This is a variational principle

$$\delta \mathscr{I} = 0. \tag{45}$$

Top and bottom boundary conditions with adjacent half-spaces are taken into account by adding suitable terms to $\mathscr{I}$ as discussed earlier (Biot, 1963, 1965).

When using the power expansion in ω^2 of the coefficients $a_{ij} b_{ij}$, the variational principle (45) yields immediately the typical form of the characteristic value problem of linear conservative mechanics with quadratic forms in U_i and V_i for the potential and kinetic energies and the corresponding Lagrangian equations.

The transfer matrices of the Thomson-Haskell method have been derived in terms of the coefficients $a_{ij} b_{ij}$ (Biot, 1963, 1965). The results remain valid in the more general case with initial and couple stresses and for orthotropic layers as explained below using appropriate values of a_{ij} and b_{ij} for each case.

Orthotropic Layers

The foregoing results for isotropic materials have been obtained directly. The same simple derivation is applicable to orthotropic layers with directions of symmetry parallel and normal to the faces. This more general case was also discussed in detail earlier in the context of an initially stressed medium. The existence of a strain energy and the symmetry with respect to the middle plane of the layer leads to a rigidity matrix with exactly the same mathematical structure as in equation (34) with six distinct coefficients. Only the values of a_{ij} and b_{ij} are different. For an orthotropic material in plane strain, the stress-strain relations are

$$\begin{aligned}\sigma_{xx} &= C_{11} e_{xx} + C_{12} e_{yy}\\ \sigma_{yy} &= C_{12} e_{xx} + C_{22} e_{yy}\\ \sigma_{xy} &= 2L e_{xy}.\end{aligned} \tag{46}$$

Substitution in the dynamical equation (3) yields

$$\begin{aligned}C_{11}\frac{\partial^2 u}{\partial x^2} + L\frac{\partial^2 u}{\partial y^2} + (C_{12} + L)\frac{\partial^2 v}{\partial x \partial y} + \omega^2 \rho u = 0\\ C_{22}\frac{\partial^2 v}{\partial y^2} + L\frac{\partial^2 v}{\partial x^2} + (C_{12} + L) + \frac{\partial^2 u}{\partial x \partial y} + \omega^2 \rho v = 0.\end{aligned} \tag{47}$$

In this case, we cannot decouple the dilatational and shear waves. For the antisymmetric case, we write a solution of the form (6) with two undetermined constants $C_1 C_2$ and roots $\beta_1 \beta_2$ of the biquadratic characteristic equation

$$\beta^4 - 2m\beta^2 + k^2 = 0 \tag{48}$$

where

$$\begin{aligned}2m &= \frac{1}{LC_{22}}\left[\Omega C_{22} - L\left(2C_{12} + \frac{\omega^2\rho}{l^2}\right) - C_{12}^2\right]\\ k^2 &= \frac{\Omega}{LC_{22}}\left(L - \frac{\omega^2\rho}{l^2}\right)\\ \Omega &= C_{11} - \frac{\omega^2\rho}{l^2}.\end{aligned} \tag{49}$$

The values of $\beta_1{}^2$ and $\beta_2{}^2$ are

$$\begin{aligned}\beta_1{}^2 &= m + \sqrt{m^2 - k^2}\\ \beta_2{}^2 &= m - \sqrt{m^2 - k^2}.\end{aligned} \tag{50}$$

Proceeding as in the case of isotropy, we derive relations (15) with the coefficients

$$a_{11} = \frac{\Omega}{\Lambda_a}(\beta_2^2 - \beta_1^2) \qquad a_{22} = \frac{C_{22}}{\Lambda_a}(\beta_2^2 - \beta_1^2) z_1 z_2$$

$$a_{12} = \frac{1}{\Lambda_a}[(\Omega + C_{12}\beta_2^2) z_1 - (\Omega + C_{12}\beta_1^2) z_2]$$

$$\Lambda_a = (\Omega - L\beta_1^2) z_2 - (\Omega - L\beta_2^2) z_1. \tag{51}$$

Values for the symmetric mode of the layer is obtained by interchanging the sinh and cosh functions. This is equivalent to replacing in (51) z_1 and z_2, respectively by $1/z_1'$ and $1/z_2'$ with

$$z_1' = \frac{1}{\beta_1}\tanh \beta_1 \gamma \qquad z_2' = \frac{1}{\beta_2}\tanh \beta_2 \gamma. \tag{52}$$

We obtain

$$b_{11} = \frac{\Omega}{\Lambda_s}(\beta_2^2 - \beta_1^2) z_1' z_2' \qquad b_{22} = \frac{C_{22}}{\Lambda_s}(\beta_2^2 - \beta_1^2)$$

$$b_{12} = \frac{1}{\Lambda_s}[(\Omega + C_{12}\beta_2^2) z_2' - (\Omega + C_{12}\beta_1^2) z_1']$$

$$\Lambda_s = (\Omega - L\beta_1^2) z_1' - (\Omega - L\beta_2^2) z_2'. \tag{53}$$

The values of a_{ij} and b_{ij} given by equations (51) and (53) are equivalent to those obtained by putting equal to zero the initial stress in the more general results derived earlier. With these values of a_{ij} and b_{ij}, the same equations (34) and (35) remain valid as well (44) with the quadratic form

$$\mathscr{I} = \sum L_i I_i \tag{54}$$

where L_i now denotes the shear modulus of the orthotropic material. Proceeding as follows, it is easily verified that these results lead to the values obtained above for the isotropic material. Comparing the stress-strain relations (2) and (46), we see that for the isotropic material

$$C_{11} = C_{22} = H \qquad C_{12} = H - 2L. \tag{55}$$

With these values, we obtain for the coefficients of the characteristic equation (48)

$$2m = 2 - \frac{\omega^2 \rho}{Hl^2} - \frac{\omega^2 \rho}{Ll^2} = \beta_1^2 + \beta_2^2$$

$$k^2 = \left(1 - \frac{\omega^2 \rho}{Hl^2}\right)\left(1 - \frac{\omega^2 \rho}{Ll^2}\right) = \beta_1^2 \beta_2^2. \tag{56}$$

Hence, the roots are

$$\beta_1^2 = 1 - \frac{\omega^2 \rho}{Hl^2} \qquad \beta_2^2 = 1 - \frac{\omega^2 \rho}{Ll^2} \tag{57}$$

which coincide with the values (8) for the isotropic case. Using the value of β_1^2 and the identity (13), we derive

$$\Omega = H\beta_1^2 \qquad \Lambda_a = (H - L)(\beta_1^2 z_2 - z_1). \tag{58}$$

Substitution of these values into (51) yields

$$a_{11} = \frac{H\beta_1^2(\beta_2^2 - \beta_1^2)}{(H - L)(\beta_1^2 z_2 - z_1)} \qquad a_{22} = \frac{H(\beta_2^2 - \beta_1^2)z_1 z_2}{(H - L)(\beta_1^2 z_2 - z_1)}$$
$$a_{12} = \frac{\beta_1^2 z_2 - \beta_2^2 z_1}{z_1 - \beta_1^2 z_2} - 1. \tag{59}$$

Again, using the identity (13), we derive

$$\frac{H}{H - L}(\beta_1^2 - \beta_2^2) = 1 - \beta_2^2. \tag{60}$$

With this value, expressions (59) of a_{ij} are the same as (16) obtained directly for the isotropic medium. A similar treatment of expressions (53) of b_{ij} also yields the values (19) for the isotropic medium. Since (51) and (53), as pointed out, are implicit in the more general results obtained earlier with initial stress, we seen how these more general results lead readily to the values (16) and (19) obtained directly for isotropic layers without initial stress.

Transverse Isotropy: Three-Dimensional Solutions Derived from Plane Strain Modes

We consider the layers to be transverse isotropic whose elastic properties are isotropic around a vertical axis normal to the layers. In any vertical plane, the stress-strain relations for the plane strain are the same as (46) for the orthotropic material. It was shown that in this case, three-dimensional solutions are immediately obtained from the plane-strain case without repeating the derivation (Biot, 1966, 1972b, 1974). For convenience, we write the plane-strain solution in the xy plane in complex form as

$$u = -iU(ly)\exp ilx \qquad \sigma_{xy} = -i\tau(ly)\exp ilx$$
$$v = V(ly)\exp ilx \qquad \sigma_{yy} = q(ly)\exp ilx. \tag{61}$$

Because of the transverse isotropy, this solution remains valid in a vertical plane which makes an angle θ with the x, y plane. We denote by x' the rotated x axis, and add a z axis normal to xy. For the rotated plane-strain solution, we obtain in three dimensions

$$u = -iU\cos\theta \exp ilx' \qquad \sigma_{xy} = -i\tau\cos\theta\exp ilx'$$
$$w = -iU\sin\theta\exp ilx' \qquad \sigma_{zy} = -i\tau\sin\theta\exp ilx'$$
$$v = V\exp ilx' \qquad \sigma_{yy} = q\exp ilx' \tag{62}$$

where u, w, and v are the displacements along xzy and

$$x' = x\cos\theta + z\sin\theta \qquad lx' = \xi x + \zeta z$$
$$\xi = l\cos\theta \quad \zeta = l\sin\theta \quad l^2 = \xi^2 + \zeta^2. \tag{63}$$

In the three-dimensional solution (62), we may also write

$$u = -iU\frac{\partial}{\partial(lx)}\exp ilx' \qquad \sigma_{xy} = -i\tau\frac{\partial}{\partial(lx)}\exp ilx'$$

$$w = -iU\frac{\partial}{\partial(lz)}\exp ilx' \qquad \sigma_{zy} = -i\tau\frac{\partial}{\partial(lz)}\exp ilx'. \tag{64}$$

By Fourier analysis, this yields a large class of the three-dimensional solutions. For example, we may multiply each solution (62) by $f(\xi, \zeta)$ and integrate over ξ and ζ. In particular, the stress σ_{yy} becomes

$$\sigma_{yy} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi, \zeta)q(ly)\exp i(\xi x + \zeta z)\, d\xi\, d\zeta. \tag{65}$$

By double Fourier transform (Sneddon, 1951), we choose $f(\xi, \zeta)$ to yield any arbitrary distribution of σ_{yy} of the normal stress at the surface of a multi-layered system with zero tangential stress ($\tau = 0$) at this surface. The values of the displacements u, w, and v at any interface are then obtained in the form (62) by solving the recurrence relations (44) for U_i and V_i as obtained for the plane-strain solution. In particular we may take the average value of the solution (62) for all orientations by simply integrating it with respect to θ from 0 to 2π with $\tau = 0$. The solution in this case is axially symmetric and u becomes the radial displacement u_r. All directions being equivalent, we put $z = 0$ and replace x by the radial distance r from the axis in expressions (62). The integration yields

$$\sigma_{yy} = \frac{q}{2\pi}\int_0^{2\pi} \exp i(lr\cos\theta)\, d\theta$$

$$v = \frac{V}{2\pi}\int_0^{2\pi} \exp i(lr\cos\theta)\, d\theta$$

$$u_r = -\frac{U}{2\pi}\frac{d}{d(lr)}\int_0^{2\pi} \exp i(lr\cos\theta)\, d\theta. \tag{66}$$

These values are (Whittaker and Watson, 1927)

$$\sigma_{yy} = qJ_0(lr)$$

$$v = VJ_0(lr)$$

$$u_r = -U\frac{dJ_0}{dr} = UJ_1(lr) \tag{67}$$

where J_0 and J_1 are the Bessel functions of the first kind of order zero and one. The case of imposed normal forces at any interface may be similarly solved in three dimensions for axially symmetric distribution using Hankel transforms (Sneddon, 1951).

Instead of using distributions such as (65), we may superpose two-dimensional solutions of different orientations but same wavenumber l. This yields various triangular, rectangular, and hexagonal patterns. The common wavenumber l corresponds to the concept of intrinsic wavelength (Biot, 1966, 1972b, 1974) of three-dimensional solutions. The resonant frequency of the pattern is the same as for the plane strain component mode of the same intrinsic wavelength.

A complete solution should also consider shear waves for which three-dimensional solutions are obtained by a similar process of superposition. Note that in the foregoing solution (64), the tangential stresses σ_{xy} and σ_{zy} are expressed as a gradients in the xz plane and may therefore not be chosen arbitrarily on a particular surface. The case of pure shear waves completes the solution by adding in the xz plane tangential stresses represented as $\sigma_{xy} = -\partial\psi/\partial z$ and $\sigma_{zy} = \partial\psi/\partial x$.

Effect of Initial Stresses, Gravity, and Couple Stresses

The theory of multi-layered media was initially developed in detail in the context of initial stresses (Biot, 1963, 1965). For transverse isotropy with principal initial stresses along the directions of symmetry, the stress-strain relations retain the same form as (46) except that the stresses are now suitably defined incremental stresses. The initial stress is made up of a term $-p_H$ which is hydrostatic and a horizontal principal stress $-P - p_H$ the same in all horizontal directions. The coefficients a_{ij} and b_{ij} remain formally the same as (51) and (53), and the initial deviatoric stress $-P$ appears only in the values of β_1 and β_2 through the characteristic equation (48) where

$$
\begin{aligned}
2m &= \frac{1}{LC_{22}}\left[\Omega C_{22} - L\left(2C_{12} + P + \frac{\omega^2\rho}{l^2}\right) - C_{12}^2\right] \\
k^2 &= \frac{\Omega}{LC_{22}}\left(L - P - \frac{\omega^2\rho}{l^2}\right).
\end{aligned}
\tag{68}
$$

The value of P may be different in each layer. The buoyancy effect of gravity is obtained by writing the recurrence equations with additional terms taking this effect into account.

$$
\frac{\partial\mathscr{T}}{\partial U_i} = 0 \qquad \frac{\partial}{\partial V_i}(\mathscr{T} + \mathscr{G}) = 0 \tag{69}
$$

where $\mathscr{T}$ is given by (54) and

$$
\mathscr{G} = \frac{1}{2l}\sum_i (\rho_{i+1} - \rho_i) g V_{i+1}^2. \tag{70}
$$

We denote by ρ_i the density of the ith layer, and g is the acceleration of gravity. Again, this result translates into the variational principle

$$
\delta(\mathscr{T} + \mathscr{G}) = 0. \tag{71}
$$

For an incompressible material, these equations express rigorously the effect of gravity forces. If this is not the case, we must take into account an additional buoyancy effect due to the change of volume. However, this effect is usually negligible in practice.

If the layers themselves are constituted by thin laminations of hard and soft sheets, a moment per unit area may appear due to the bending rigidity of the laminations. This moment is proportional to the local curvature and is written as (Biot, 1972a)

$$\mathscr{M} = b \frac{\partial_v^2}{\partial x^2}.$$

We have shown that this effect may be included very simply replacing P by $P - bl^2$ in the values (68). This property was referred to as the couple stress analogy (Biot, 1974). The superposition of plane-strain modes to derive three-dimensional solutions as previously described is, of course, valid for the more general problem with initial and couple stresses. In particular, in problems of instability, we have shown how these plane-strain solutions along directions at 120° yield a pattern of hexagonal cells of the Bénard type.

Viscoelastic Layers, Linear Thermodynamics, and Basic Reciprocity Relations

The mechanics of viscoelastic media initially in mechanical and thermodynamic equilibrium, whether initially stressed or not, may be considered from the standpoint of linear thermodynamics. Small perturbations obey Onsager's reciprocity relations and are governed by linear Lagrangian equations derived from a kinetic and potential energy and a dissipation function. Transfer impedances of such a system obey fundamental reciprocity relations similar to those of a black box of electric circuits with inductance resistance and capacitance. The internal behavior is represented by internal coordinates which determine the frequency dependence and heredity properties of the complex impedances of the outlets. In viscoelasticity, this leads to a correspondence principle (Biot, 1965) whereby the complex coefficients of the stress-strain relations exhibit the same basic symmetry and reciprocity properties as the elastic moduli. Hence, the four moduli of the stress-strain relations (46) become complex functions of the frequency with the same equality of the off-diagonal terms in the matrix. Also, the type of frequency dependance is determined by the fundamental thermodynamics. This leads to a 4×4 impedance matrix for the viscoelastic layer as well as complex coefficients a_{ij} and b_{ij} with exactly the same reciprocity relations and mathematical structure as the rigidity matrix of equation (34). A detailed discussion of these fundamental properties as well as a large number of references will be found in the authors monograph (Biot, 1965). In stability problems, $i\omega$ is replaced by p. A fundamental theorem states that the characteristic equation for unstable solutions always yields real and positive values of p.

Conclusions

It has been shown how stiffness matrices for multi-layered media may be derived by a very simple and direct procedure for isotropic and orthotropic layers. As a consequence of the procedure, the matrices obtained are real, symmetric, and dimensionless. They contain only six simple distinct elements leading to simplified programming. They also provide physical insight and their structure belongs to a universal type which follows from fundamental principles of linear conservative mechanics. Decoupling of the two faces of a layer for large thickness to wavelength ratio becomes self-evident leading to half-space matrices. General three-dimensional solutions for transverse isotropy are immediate by a process of superposition of plane strain modes with the same plane-strain matrices. The matrices are valid for viscoelastic materials as a consequence of fundamental principles of linear thermodynamics. The procedure followed here is the same as developed many years ago by the author for initially stressed multi-layered media with anisotropy and couple stresses, and it is shown how the earlier results lead explicitly to those obtained here as a particular case.

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