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Simplified stability criteria for nonconservative dynamical systems with application to wing flutter

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Simplified stability criteria for nonconservative dynamical systems with application to wing flutter (*)

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Abstract. — Oscillations of a purely gyrostatic system with two degrees of freedom are constituted by elliptic precessions rotating in opposite directions. Addition of nonconservative forces feeds power into one of them, generating instability. The effect of damping provides a stability condition whose advantage and simplicity over Routh's criterion is indicated. A gyroscopic analog model has been built and is described. Application to wing flutter is briefly discussed. Generalization to *n* degrees of freedom is immediate. The theory involves second order perturbation.

1. INTRODUCTION

A linear dynamical system is considered which includes damping, gyrostatic and nonconservative forces. We first consider the two precessions of the purely gyrostatic system with two degrees of freedom. The work done by nonconservative forces on these precessions is evaluated and compared to the energy dissipated. This provides a stability criterion showing that only one of these precessions can be unstable. By applying Routh's stability criterion it is found that it may be factorized, each factor corresponding to one of the two precessions and leading to exactly the same stability criterion as derived here by energy balance. In addition to its simplicity the latter has the advantage of determining which particular precession is unstable while Routh's criterion is global.

A gyroscopic model illustrating these properties has been built and is described. The model constitutes an analog for wing flutter and a short discussion provides physical insight clarifying well known fea-

(*) This work was developed by the author at the California Institute of Technology in the year 1940 and is presented here for the first time.

tures derived empirically from numerical analysis. The results are readily generalized to n degrees of freedom.

2. Oscillating gyrostatic system

A two degree of freedom system with elastic restraint and gyrostatic forces is governed by the equations

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 + \gamma \dot{q}_2 &= 0 \\ \ddot{q}_2 + \omega_2^2 q_2 - \gamma \dot{q}_1 &= 0 \end{aligned} \tag{1}$$

The dot above a symbol denotes a time derivative. The system is described by generalized coordinates q_1 and q_2, and γ is a coefficient defining the gyrostatic forces. For $\gamma = 0$ there are two uncoupled natural modes of angular frequencies ω_1 and ω_2.

When $\gamma \neq 0$ the amplitudes q_1, q_2 are not independent. The solutions are of the type exp(pt) where p satisfies the characteristic equation

$$p^4 + (\omega_1^2 + \omega_2^2 + \gamma^2)p^2 + \omega_1^2\omega_2^2 = 0 \tag{2}$$

The roots p^2 are real and negative and the roots p are pure imaginary $i\Omega_1$, $i\Omega_2$. For small values of γ these roots differ from $i\omega_1$ and $i\omega_2$ by a term proportional to γ^2. Hence we may write approximately $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$. In the corresponding solutions the values of q_1 and q_2 are 90 degrees out of phase. The solution of frequency ω_1 is

$$\begin{aligned} q_1 &= M_1 \cos(\omega_1 t + \alpha_1) \\ q_2 &= M_1 \frac{\gamma\omega_1}{\omega_1^2 - \omega_2^2} \sin(\omega_1 t + \alpha_1) \end{aligned} \tag{3}$$

where M_1 and α_1 are arbitrary. Similarly the solution of frequency ω_2 is

$$\begin{aligned} q_1 &= M_2 \frac{\gamma\omega_2}{\omega_1^2 - \omega_2^2} \sin(\omega_2 t + \alpha_2) \\ q_2 &= M_2 \cos(\omega_2 t + \alpha_2) \end{aligned} \tag{4}$$

We shall assume $\omega_1 > \omega_2$.

These solutions represent elliptic precessions in the plane q_1, q_2. In solution (3) the representative point describes a flat ellipse with a higher frequency ω_1 and its large axis is along q_1. In the second solutions (4) of lower frequency ω_2 the large axis lies along q_2. The two types of solutions are obtained from each other by interchanging q_1 and q_2. Hence *the two precessions rotate in opposite directions.* This feature as we shall see has an important bearing on stability properties.

3. Instability due to nonconservative forces

With the addition of nonconservative forces equations (1) are written

$$\begin{aligned} \ddot{q} + \omega_1^2 q_1 + \gamma \dot{q}_2 &= -kq_2 \\ \ddot{q}_2 + \omega_2^2 q_2 - \gamma \dot{q}_1 &= kq_1 \end{aligned} \tag{5}$$

The nonconservative terms on the right side are assumed small and represent a force normal to the radius vector $(q_1\ q_2)$ acting clockwise or counterclockwise depending on the sign of k. It performs work on the two precessions (3) and (4). Since they rotate in opposite directions this work is positive for one precession and negative for the other. Hence the *nonconservative force produces instability of one of the precessions and damps out the other.*

The average power input of the nonconservative force on the precession over a time interval t covering an integer number of cycles is

$$\mathrm{P}^{av} = \frac{1}{t} \int_0^t (kq_1\dot{q}_2 - kq_2\dot{q}_1)dt \tag{6}$$

Substituting the values (3) and (4) we find

$$\mathrm{P}_1^{av} = \mathrm{M}_1^2 \frac{\omega_1^2}{\omega_1^2 - \omega_2^2} k\gamma \qquad \mathrm{P}_2^{av} = -\mathrm{M}_2^2 \frac{\omega_2^2}{\omega_1^2 - \omega_2^2} k\gamma \tag{7}$$

where P_1^{av} is the power input into the precession of higher frequency ω_1 and P_2^{av} the power input into the precession of lower frequency ω_2. We note that the stability depends on the sign of $k\gamma$. If it is positive it is the higher frequency precession which is unstable.

This result brings to light the important feature that the *instability is proportional to $k\gamma$ and inversely proportional to the difference of the squares of the natural frequencies of the system.*

4. Stability criteria based on energy balance

Let us add damping to the system. Introducing a dissipation function D, the power dissipated is

$$2D = \beta_1 \dot{q}_1^2 + 2\beta_{12}\dot{q}_1\dot{q}_2 + \beta_2\dot{q}_2^2 \tag{8}$$

and the equations of motion (5) become

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 + \beta_1\dot{q}_1 + \beta_{12}\dot{q}_2 + \gamma\dot{q}_2 &= -kq_2 \\ \ddot{q}_2 + \omega_2^2 q_2 + \beta_{12}\dot{q}_1 + \beta_2 q_2 - \gamma\dot{q}_1 &= kq_1 \end{aligned} \tag{9}$$

The average power dissipated over a time t is expressed by

$$2D^{av} = \frac{2}{t}\int_0^t D\,dt \tag{10}$$

where t covers an integer number of cycles.

In the precession of frequency ω_1 the main motion is along q_1 while for the other precession the main motion is along q_2.

Hence we derive

$$2D_1^{av} = \frac{1}{2} M_1^2 \beta_1 \omega_1^2 \qquad 2D_2^{av} = \frac{1}{2} M_2^2 \beta_2 \omega_2^2 \tag{11}$$

where $2D_1^{av}$ and $2D_2^{av}$ represent respectively the power dissipated in the precessions of frequencies ω_1 and ω_2.

Stability requires that the dissipation be larger than the power input. This requires

$$2D_1^{av} > P_1^{av} \qquad 2D_2^{av} > P_2^{av} \tag{12}$$

Substitution of the values (7) and (11) yields the stability criteria

$$\beta_1 > \frac{2k\gamma}{\omega_1^2 - \omega_2^2} \qquad \beta_2 > -\frac{2k\gamma}{\omega_1^2 - \omega_2^2} \tag{13}$$

Again depending on the sign of $k\gamma$ one of them is always identicaly verified and we need only satisfy the other.

5. Comparison with Routh's stability criterion

The differential equations (9) represent the most general linear two degree of freedom dynamical system with elastic restraint and constant coefficients. Its general solutions is expressed by characteristic solutions of the form exp(pt) where p is a root of the characteristic equation

$$p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0 \tag{14}$$

with

$$\begin{aligned} A_0 &= \omega_1^2\omega_2^2 + k^2 \\ A_1 &= \beta_1\omega_2^2 + \beta_2\omega_1^2 + 2k\gamma \\ A_2 &= \omega_1^2 + \omega_2^2 + \beta_1\beta_2 - \beta_{12}^2 + \gamma^2 \\ A_3 &= \beta_1 + \beta_2 \end{aligned} \tag{15}$$

The system is stable if all characteristic roots have a negative real part. The necessary and sufficient condition for this to be the case is that all coefficients (15) be positive and in addition that

$$A_1A_2A_3 - A_3^2A_0 - A_1^2 > 0 \tag{16}$$

This is known as Routh's stability criterion, (see [1]). In our case it is immediately evident that A_0 A_2 A_3 are always positive so that we are left with the two conditions $A_1 > 0$ and (16).

If γ and k as well as the damping are small the coefficients (15) become

$$\begin{aligned} A_0 &= \omega_1^2\omega_2^2 \\ A_1 &= \beta_1\omega_2^2 + \beta_2\omega_1^2 + 2k\gamma \\ A_2 &= \omega_1^2 + \omega_2^2 \\ A_3 &= \beta_1 + \beta_2 \end{aligned} \tag{17}$$

We substitute these values into expression (16). A large number of terms cancel out and the remaining ones may be factorized in the following form

$$[\beta_1(\omega_1^2 - \omega_2^2) - 2k\gamma][\beta_2(\omega_1^2 - \omega_2^2) + 2k\gamma] > 0 \tag{18}$$

On the other hand the second criterion $A_1 > 0$ may be written

$$A_1(\omega_1^2 = \omega_2^2) = \omega_2^2[\beta_1(\omega_1^2 - \omega_2^2) - 2k\gamma] + \omega_1^2[\beta_2^2(\omega_1^2 - \omega_2^2) + 2k\gamma] > 0 \tag{19}$$

The inequality (18) shows that the factors must be of the same sign and inequality (19) that they must be positive.

Hence Routh's stability criterion becomes

$$\beta_1(\omega_1^2 - \omega_2^2) - 2k\gamma > 0 \qquad \beta_2(\omega_1^2 - \omega_2^2) + 2k\gamma > 0 \tag{20}$$

This coincides exactly with the stability criteria (13) derived from energy balance. It can be seen that they are much simpler that Routh's criterion in its original form. Moreover they indicate which precession is unstable while Routh's criterion refers only to global instability.

6. Demonstration model ([1])

A mechanical model governed by equations (5) has been built for the purpose of demonstrating the precessional instability. It is shown in figure 1. An electric motor EM with a vertical axis A is suspended through springs S from an inverted U frame F itself fixed to a base B. A horizontal copper disk D is mounted at the bottom of the motor shaft. The disc rotates above a horizontal horse-shoe magnet of poles P. There are two degrees of freedom measured by the angle q_1 of oscillation of the axis about a center O in the plane of the figure and an angle q_2 of oscillation normally to this plane. The natural frequencies of these motions are ω_1 and ω_2 with $\omega_1 > \omega_2$. When the motor rotates the disc D introduces gyrostatic forces $\gamma\dot{q}_2$ and $-\gamma\dot{q}_1$ in equations (5). Also when the disc is off center the braking effect of the eddy currents induced by the magnet generates a nonconservative force normal to the displacement from the center. This force is represented by the terms $-kq_2$ and kq_1 in equations (5). In order to predict the behaviour of the model consider the case where an observer standing on the disc witnesses a rotation from right to left. In this case γ and k are both positive. Hence $k\gamma$ is positive. Note that a change of

([1]) The model was built in the year 1940 at the California Institute of Technology and has been recently donated to the Institute of Applied Mechanics of the Free University of Brussels.

the direction of rotation reverses the sign of both γ and k so that $k\gamma$ remains positive. As a consequence the precession of higher frequency ω_1, is always the unstable one. This is in fact what is observed. The axis exhibits a tilting oscillation q_1 involving the springs S and the axis A performs a flat elliptic precession whose major axis lies in the plane of the figure.

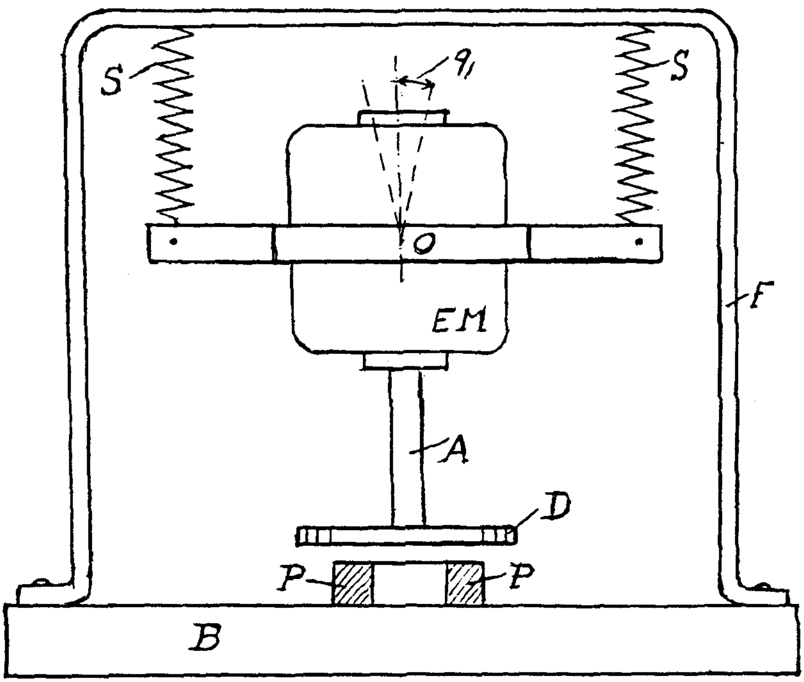

FIG. 1. — *Demonstration Model:* EM electric motor suspended throught springs S form a frame F fixed to a base B · A copper disc D, attached to the bottom of the motor shaft A, rotates above a horseshoe magnet of poles P. When the motor rotates it oscillates in a flat elliptic conical precession, of major axis in the plane of the figure, and tilting motion q_1 in that plane.

7. APPLICATION TO AEROELASTIC WING FLUTTER

The foregoing results provide a fundamental simple model for the unstable oscillations of a wing of infinite span facing a wind of velocity V a phenomenon known as flutter. The wing is restrained elastically in pitching motion and vertical translation. The aerodynamic forces acting on the oscillating wing have been derived by Theodorsen and involve the so-called Theodorsen function Y which is a complex quantity dependent on $\omega c/2V$ (ω angular frequency, c wing chord) (see [2]). In the practical range we may put approximately $Y = \frac{1}{2}$. With this value it is possible to show that the equations of oscillation of the wing may be written in the form (9) with $k\gamma$ positive. The generalized coordinates of the system are q_1 which represents mainly a rotational mode of higher frequency ω_1 and q_2 which represents mainly a translational mode of lower frequency ω_2. These modes which are natural modes involve masses which include the virtual mass of the surrounding air and certain aerodynamic forces which act like elastic force and are q_1 dependent.

The foregoing conclusions regarding equations (9) are applicable here. Since $k\gamma > 0$ the *rotational oscillation* of higher frequency ω_1 is the unstable one and the stability condition is given by (13) as

$$\beta_1 > \frac{2k\gamma}{\omega_1^2 - \omega_2^2} \tag{21}$$

where β_1 is the aerodynamic damping of the rotational oscillation. We note the increase of instability for a decreasing difference between frequencies. It can also be shown that moving the center of gravity forward increases β_1 hence improves the stability. When the wind velocity V increases β_1 and γ are proportional to V while k is proportional to V^2. Hence $k\gamma$ is proportional to V^3. As a consequence the inequality (21) shows the sudden explosive appearance of flutter as the wind velocity increases beyond a critical value. This effect is further enhanced by the fact that ω_1^2 also tends to decrease with increasing V.

The fundamental characteristics of flutter derived here quite simply from the analog gyroscopic model are well known to flutter specialists as empirical rules derived from numerical analysis.

We have assumed $Y = \frac{1}{2}$ but similar qualitative conclusions may be obtained by introducing other values of Y including complex ones.

8. Generalization to n degrees of freedom

With n degrees of freedom equations (9) are generalized to

$$\ddot{q}_i + \omega_i^2 q_i + \sum_j \beta_{ij}\dot{q}_j + \sum_j \gamma_{ij}\dot{q}_j = \sum_j k_{ij} q_j \tag{22}$$

with

$$\beta_{ij} = \beta_{ji} \qquad \gamma_{ij} = -\gamma_{ij} \qquad k_{ij} = -k_{ij}$$

Without damping and nonconservative forces the equations are reduced to those of a purely gyrostatic system

$$\ddot{q}_i + \omega_i^2 q_i + \sum_j \gamma_{ij}\dot{q}_j = 0 \tag{23}$$

General solutions of the type exp(pt) leads to an algebraic equation for the characteristic roots p of degree $2n$. However the equation remains invariant if we change p into $-p$ hence it is an equation of degree n in p^2. It is easy to show that these roots are real and negative. Hence the roots p are pure imaginary $\pm\Omega i$. Also the characteristic equation for p^2 is invariant if we replace γ_{ij} by $-\gamma_{ij}$. Hence if γ_{ij} is of the first order the roots Ω differ from ω_i by terms of the second order so that we may put the Ω's equal to ω_i.

As a consequence each characteristic solution of (23) involves q_i and a set of q_j ($j \neq i$) of the order of the γ_{ij}'s. Neglecting higher order terms we write (23) as $n - 1$ equations

$$\ddot{q}_j + \omega_j^2 q_j + \gamma_{ji}\dot{q}_i = 0 \tag{24}$$

For any particular characteristic solution of approximate frequency ω_i this determines q_j in terms of q_i. If we put

$$q_i = \mathrm{M}_i \cos(\omega_i t + \alpha_i) \tag{25}$$

equations (24) yield

$$q_j = \frac{\mathrm{M}_i \gamma_{ji} \omega_i}{\omega_i^2 - \omega_j^2} \sin(\omega_i t + \alpha_i) \qquad j \neq i \tag{26}$$

This generalizes the two-dimensional solutions (3) and (4). For each axis i expressions (26) define a direction q_j in the $(n - 1)$ dimensional plane $q_i = 0$. This direction along with q_i defines a plane in the n space and equations (25) (26) represent an elliptic precession in this plane with a large axis along q_i.

There are n such elliptic precessions representing the general solution. In order to derive the stability of any particular precession we evaluate the average power input of the nonconservative force on the precession. This is written in general

$$P^{av} = \frac{1}{t}\int_0^t \sum_{\mu\nu} k_{\mu\nu} q_\mu \dot{q}_\nu dt \qquad (27)$$

We substitute the values (25-26) for the i^{th} precession and retain only the terms containing q_i since the others are of higher order. This yields

$$P_i^{av} = \frac{1}{t}\int_0^t \sum_j (k_{ij} q_i \dot{q}_j + k_{ji} q_j \dot{q}_i) dt = M_i^2 \omega_i^2 \sum_j \frac{k_{ij}\gamma_{ji}}{\omega_i^2 - \omega_j^2} \qquad (28)$$

The energy dissipated by the i^{th} precession is

$$2D_i^{av} = \frac{1}{2}\beta_{ii} M_i^2 \omega_i^2 \qquad (29)$$

Stability of the precession requires that this energy dissipated be larger than P_i^{av}. Hence the stability criterion

$$\beta_{ii} > \sum_j \frac{2k_{ij}\gamma_{ji}}{\omega_i^2 - \omega_j^2} \qquad (30)$$

which generalizes the criterion (13) to n degrees of freedom and determines which of the n precessions are unstable. It is valid only if $\omega_i \neq \omega_j$.

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