Temperature Analysis in Hydraulic Fracturing

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Summary. We present a thermal analysis of hydraulic fracturing based on variational methods. Our purpose is to provide a theoretical method for determining fracturing fluid temperature as a function of time and location during fracture growth. We first develop an expression of the variational principle for the general problem of convective heat transfer in a porous solid. Its accuracy is confirmed by comparisons with exact relations for specific cases. It is then used to develop a partial-differential equation for fluid temperature as a function of time and location. In this development, we treat fracture dimensions and leakoff distribution as known functions. The differential equation is solved by the method of characteristics. An alternative method of successive approximations is also presented. This solution can be combined iteratively with a fracture propagation analysis to find self-consistent results for fracture dimensions, leakoff, and temperature. We consider results obtained this way with those obtained with the two-dimensional (2D) Lagrangian analysis. Results are presented as profiles of temperature vs. dimensionless fracture length. When these profiles are normalized in terms of reservoir temperature and wellbore temperature, they change little with time of treatment, fracturing conditions, or reservoir properties. A simple profile with two straight-line segments is a good approximation for most fracturing treatments. This approximate profile is very useful for field operations. It provides a way to estimate fracturing temperatures rapidly during a treatment.

Introduction

Hydraulic fracturing technology includes an abundance of theoretical work on the problem of fracture propagation. Theories have evolved from the simplest 2D analyses to modern three-dimensional (3D) computer models. In this evolution, much attention has been given to the problems of fracture mechanics, rock deformation, fluid leakoff, proppant transport, and fluid rheology. The equally important problem of fluid temperature distributions in the fracture has received little attention.

Temperature considerations are especially important in modern fracturing operations where gelled fluids are used almost exclusively. These non-Newtonian gels have viscosity and sand-carrying characteristics that are very temperature-sensitive. At critical temperatures, they begin to decompose and lose virtually all their sand-carrying capabilities. Extending these decomposition limits to higher temperatures has been an area of intensive research recently. Exploiting these efforts requires a confident knowledge of fluid temperatures in the fracture during the treatment.

Thus, the problem of fracture temperature is a practically important one. Nevertheless, theoretical contributions in this area have been few in number. Best known is the important paper of Whitsitt and Dysart,[1] who used energy- and mass-balance principles to derive a relationship for temperature in a propagating crack as a function of location and time. Although this was a pioneering work, it has certain shortcomings. It relies on a complicated Laplace transform that is not strictly applicable. It overlooks some essential features of the problem, and its final result is not very adaptable to finding self-consistent values for crack dimensions and temperatures.

Additional contributions have been made by Wheeler,[2] by Sinclair,[3] and by Poulsen and Lee.[4] Wheeler considered the problem of heat transfer by steam injected into an existing fracture. Because the fracture was treated as static, the results say little about temperature changes in a propagating fracture. Sinclair applied Wheeler's results to a growing fracture but treated fracture propagation independently of heat transfer. Because the two are strongly coupled, this approach is not likely to produce realistic results. Poulsen and Lee suggested some modifications to Whitsitt and Dysart's method.

Variational methods provide a much more natural approach to the fracture temperature problem. Such methods have already been developed for heat conduction and convection in mixed solid/fluid systems.[5] It remains only to extend these general methods to the problem of crack growth with leakoff.

This is the approach we have used. Our purpose is to provide a detailed analysis of fluid temperature as a function of time and location during fracture growth with leakoff. As a starting point, we assume that the time history of crack growth has been determined by a method that neglects temperature effects. Thus, crack dimensions and leakoff are treated as known functions in the thermal analysis.

A second analysis of crack growth gives a better approximation by including temperature effects found from the thermal analysis. This process is repeated until convergence is obtained. The final results for crack dimensions and temperature distribution are thus self-consistent.

The thermal analysis developed this way is general enough to be used with almost any crack propagation theory. We include in this paper an example of its use with the 2D Lagrangian analysis. We assume that wellbore temperature at the fracture entrance is known from an independent analysis, such as that given by Ramey.[6]

We compare results obtained from our variational approach with exact solutions for specific cases to verify the general accuracy of the method. The accuracy has also been improved by appropriate adjustments of constants.

The variational approach leads to a general partial-differential equation for temperature in the fracture as a function of time and position. We present two methods of solution: one based on successive approximations and the other on the method of characteristics. The latter is used in our examples on the basis of Lagrangian propagation analysis.

Variational Principle

General variational principles for a porous solid have already been developed.[5] We wish to apply these principles to the problem of deriving temperature distributions in a propagating fracture.

We treat the problem of heat flow away from the fracture face by considering a porous half-space. We take the y axis normal to the plane boundary, $y=0$, as in Fig. 1. The positive direction is toward the porous medium. We assume that fluid leakoff can be represented by a uniform volumetric fluid velocity, v, parallel to the y axis.

*Deceased.

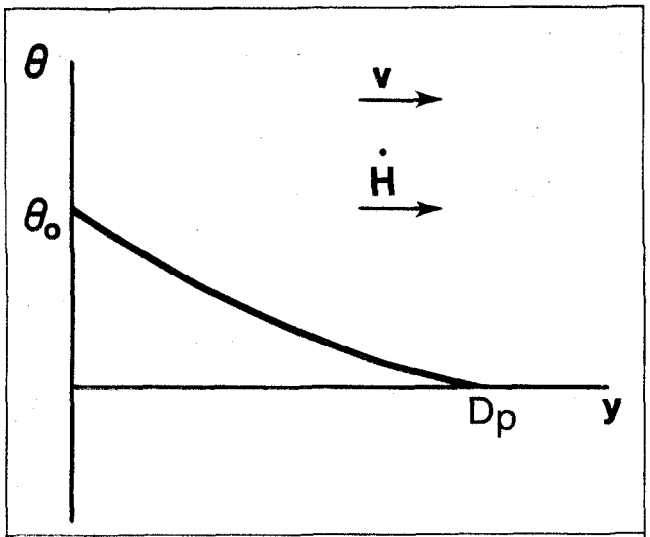

Fig. 1—Two-dimensional representation of heat flow with leakoff.

We consider $\dot{H}$, the rate of heat flow per unit area of porous solid in the y direction. $\dot{H}$ includes heat flow by both conduction and convection. The area in this case includes both the solid matrix and the pores. The y component of $\dot{H}$, as well as the temperature, θ, are assumed to be functions of y only.

We can take $\dot{H}$ to be the sum of a convective and a conductive term,

$$\dot{H}=J+vC_f\theta. \quad \ldots\ldots (1)$$

The conductive term, J, can be expressed as

$$J=-k_m\frac{d\theta}{dy}, \quad \ldots\ldots (2)$$

where k_m is the mixed thermal conductivity of the fluid-filled porous medium and can be expressed approximately as

$$k_m=\phi k_f+(1-\phi)k_s. \quad \ldots\ldots (3)$$

Conservation of energy requires that

$$C_m\dot{\theta}=-\frac{d\dot{H}}{dy}. \quad \ldots\ldots (4)$$

C_m is the mixed heat capacity of the fluid-filled porous medium. We can write it approximately as

$$C_m=\phi C_f+(1-\phi)C_s. \quad \ldots\ldots (5)$$

We integrate Eq. 4 with respect to time under the initial conditions $\theta=H=0$ at $t=0$. This gives

$$C_m\theta=-\frac{dH}{dy}. \quad \ldots\ldots (6)$$

To derive the variational principle, we follow the procedure in Ref. 1. We multiply Eq. 2 by the variation δH and integrate with respect to y:

$$\int_0^{D_p}\left(\frac{1}{k_m}J\delta H+\frac{d\theta}{dy}\delta H\right)dy=0. \quad \ldots\ldots (7)$$

Appendix A shows how this result leads to the following expression of the variational principle:

$$\frac{1}{k_m}\int_0^{D_p}(\dot{H}-vC_f\theta)\frac{\partial H}{\partial D_p}dy+\frac{\partial V}{\partial D_p}=Q, \quad \ldots\ldots (8)$$

with

$$Q=\theta_0\frac{\partial H_0}{\partial D_p}. \quad \ldots\ldots (9)$$

and

$$V=\frac{C_m}{2}\int_0^{D_p}\theta^2dy. \quad \ldots\ldots (10)$$

Relationship Between Temperature and Heat Flow

We use Eq. 8 to derive a relationship between temperature, θ_0, and the rate of heat flow, $\dot{H}_0$.

We assume a linear distribution of temperature, θ, between the surface $y=0$ and the point $y=D_p$, where $\theta=0$:

$$\theta=\theta_0\left(1-\frac{y}{D_p}\right). \quad \ldots\ldots (11)$$

Within certain limits, the surface temperature, θ_0, may be time-dependent. The penetration depth, D_p, is an unknown function of time to be determined.

In Appendix B, we use Eq. 11 to derive expressions for each of the terms in Eq. 8. These include the important relation

$$\dot{H}_0=\tfrac{1}{2}C_m(\dot{D}_p\theta_0+D_p\dot{\theta}_0). \quad \ldots\ldots (12)$$

Substituting the relations of Appendix B into Eq. 8 gives the following expression of the variational principle:

$$2D_p\dot{D}_p-\frac{25vC_fD_p}{8C_m}+\frac{9\dot{\theta}_0D_p^2}{8\theta_0}=5\frac{k_m}{C_m}. \quad \ldots\ldots (13)$$

In Appendix C, we evaluate the accuracy of these results by considering some particular cases for which the correct results are known. This exercise leads to correction and simplification of Eqs. 12 and 13. The final results are

$$2D_p\dot{D}_p-\frac{4vC_fD_p}{C_m}=5\frac{k_m}{C_m} \quad \ldots\ldots (14)$$

and

$$\dot{H}_0=\tfrac{1}{2}C_m\dot{D}_p\theta_0+0.2736C_mD_p\dot{\theta}_0. \quad \ldots\ldots (15)$$

Differential Equations for Temperature Distributions

Eqs. 14 and 15 provide a time-dependent relationship between the surface heat flow, $\dot{H}_0$, and temperature, θ_0. We use this relationship to derive the differential equation for the temperature of the fluid, $\theta_0(x,t)$, at point x and time t. We take x to be the coordinate along the crack, as shown in Fig. 2.

The width distribution of the crack is

$$w=w(x,t). \quad \ldots\ldots (16)$$

We take v to be the volumetric velocity of fluid per unit area flowing because of leakoff at point x and time t:

$$v=v(x,t). \quad \ldots\ldots (17)$$

The volumetric rate of flow in the crack across the width, w, at point x and time t is

$$q=q(x,t). \quad \ldots\ldots (18)$$

As the crack propagates, the quantities w,v, and q vary with x and t. We assume that this dependence has been evaluated in a preceding crack propagation analysis that neglects temperature effects. By this analysis, we can determine the time, t_0, at which the crack tip reaches the point x. This time, of course, is a function of x:

$$t_0=t_0(x). \quad \ldots\ldots (19)$$

We assume that the fluid is incompressible. We can write the following expression for the conservation of its mass:

$$\frac{\partial w}{\partial t}+\frac{\partial q}{\partial x}+2v=0. \quad \ldots\ldots (20)$$

Also, we can write

$$q=\int_x^L\left(\frac{\partial w}{\partial t}+2v\right)dx. \quad \ldots\ldots (21)$$

We neglect heat conduction in the x direction, in both the crack and the porous medium. Then the relationship between H_0 and θ_0 in the crack is given by a one-dimensional analysis along y as derived in the preceding section.

These assumptions lead to the following relationship for conservation of energy in the fluid within the crack:

$$\frac{\partial}{\partial x}(\theta_0 C'q)+2\dot{H}_0+\frac{\partial}{\partial t}(\theta_0 C'w)=0. \quad \ldots\ldots (22)$$

We note that the heat capacity, C', of fluid in the crack can be different from that in the porous medium.

In Appendix D, we show how Eq. 22 leads to the result

$$C'q\frac{\partial\zeta}{\partial x}+(0.5472C_mD_p+C'w)\frac{\partial\zeta}{\partial t}+2v(C'-C_f)-\frac{5k_m}{2D_p}=0. \quad \ldots\ldots (23)$$

This is the required differential equation in θ_0 expressed in terms of the new variable, ζ, defined by Eq. D-5. When the heat capacity of the fluid is the same in the crack and in the pores, it reduces to

$$C'q\frac{\partial\zeta}{\partial x}+(0.5472C_mD_p+C'w)\frac{\partial\zeta}{\partial t}-\frac{5k_m}{2D_p}=0. \quad \ldots\ldots (24)$$

In this case, the effect of convection in the pores appears only through D_p.

Methods of Solution

We must solve Eqs. 14 and 23 for D_p and ζ. Because Eq. 14 does not contain ζ, it is easily solved for D_p, provided that the form of v is known.

We can assume the approximate form of v to be[7]

$$v(x,t)=\frac{\alpha(x)}{\sqrt{t-t_0(x)}} \quad \text{for} \quad t>t_0 \quad \ldots\ldots (25a)$$

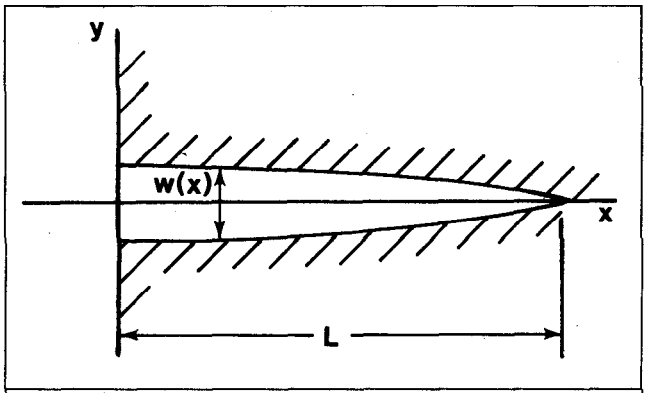

Fig. 2—Two-dimensional representation of the fracture.

and

$$v(x,t)=0 \quad \text{for} \quad t<t_0. \quad \ldots\ldots (25b)$$

Taking the initial conditions, $D_p=0$ at $t=t_0$ then gives a solution for Eq. 14 of the form

$$D_p(x,t)=C_1\sqrt{t-t_0(x)}. \quad \ldots\ldots (26)$$

Substitution of Eq. 26 into Eq. 14 gives a quadratic equation in C_1, which yields

$$C_1(x)=\frac{2\alpha C_f}{C_m}+\sqrt{\frac{4\alpha^2C_f^2}{C_m^2}+5\frac{k_m}{C_m}}. \quad \ldots\ldots (27)$$

Substituting Eq. 26 into Eq. 23 gives an equation with only one unknown, ζ.

Boundary conditions are determined as follows. If we assume that the fluid at the crack entrance is at constant temperature, we may write

$$\theta_0(0)=-\theta_f, \quad \ldots\ldots (28)$$

where θ_f is the amount by which the formation temperature exceeds the fluid temperature at the crack entrance. If we choose the reference temperature $\theta_r=\theta_f$, we find

$$\zeta(x,t)=\log\left(-\frac{\theta_0}{\theta_f}\right) \quad \ldots\ldots (29a)$$

and

$$\zeta(0,t)=0. \quad \ldots\ldots (29b)$$

We consider two methods of solving Eq. 23. One is a method of successive approximations. We find an initial approximate solution by neglecting the term in $\dot{\zeta}$. In doing so, we make two approximations. One is replacing the constant 0.2736 by zero, which is equivalent to neglecting the influence of $\dot{\theta}_0$ on $\dot{H}_0$ in Eq. 15. The other is neglecting the variation $C'w\dot{\zeta}$ of heat content of fluid in the crack. With these approximations, Eq. 23 becomes

$$C'q\frac{\partial\zeta}{\partial x}=2v(C_f-C')+\frac{5k_m}{2D_p}. \quad \ldots\ldots (30)$$

Integration gives

$$\zeta=\int_0^x\frac{1}{C'q}\left[2v(C_f-C')+\frac{5k_m}{2D_p}\right]dx. \quad \ldots\ldots (31)$$

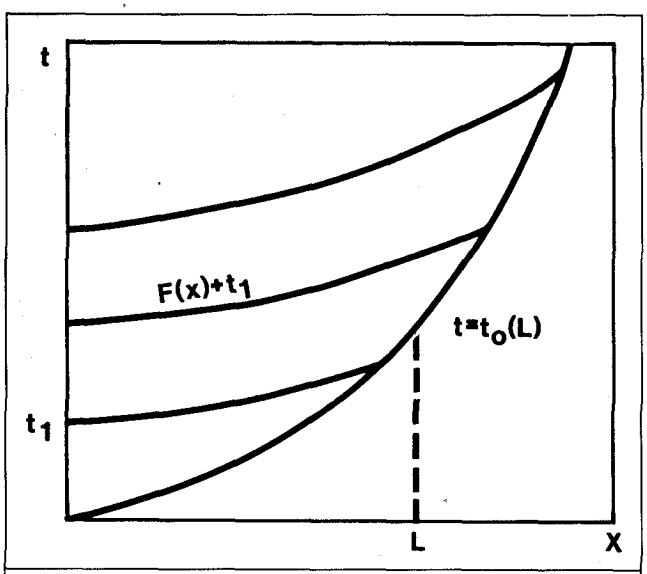

Fig. 3—Family of curves given by Eq. 37.

We then introduce this approximate ζ into Eq. 23 through $\dot{\zeta}$ and integrate again with respect to x. This procedure is repeated until we obtain convergence.

The other method of solution for Eq. 23 is the classic method of characteristics. This procedure is applied in the 3D $x/t/\zeta$ space. The method is carried out as follows.

We rewrite Eq. 23 in the form

$$P(x,t)\frac{\partial\zeta}{\partial x}+T(x,t)\frac{\partial\zeta}{\partial t}=R(x,t), \quad \ldots\ldots (32)$$

where we have taken

$$P(x,t)=C'q, \quad \ldots\ldots (33a)$$

$$T(x,t)=0.5472C_mD_p+C'w, \quad \ldots\ldots (33b)$$

and

$$R(x,t)=\frac{5k_m}{2D_p}-2v(C'-C_f). \quad \ldots\ldots (33c)$$

The characteristics in the x,t,ζ space are curves defined by the differential equations

$$\frac{dx}{P}=\frac{dt}{T}=\frac{d\zeta}{R}. \quad \ldots\ldots (34)$$

There is a two-parameter family of such curves. The solution $\zeta(x,t)$ of Eq. 23 that satisfies the boundary condition $\zeta(0,t)$ is a surface constituted by a one-parameter family of characteristics. All go through the lines $x=\zeta=0$ and hence through the t axis.

To derive the surface $\zeta(x,t)$, we first integrate the equation

$$\frac{dx}{P}=\frac{dt}{T} \quad \ldots\ldots (35)$$

or

$$\frac{dt}{dx}=\frac{T(x,t)}{P(x,t)}. \quad \ldots\ldots (36)$$

We write the solution of this first-order differential equation in the x,t plane as

$$t-t_1=F(x). \quad \ldots\ldots (37)$$

This is a one-parameter family of curves in which the parameter is the constant of integration, t_1. It is chosen so that $F(0)=0$; hence $t=t_1$ at $x=0$.

The family of curves (Eq. 37) is the projection of the characteristics on the x,t plane. The curves are illustrated in Fig. 3. They lie within the boundary defined by the curve $t=t_0(L)$.

Next, we integrate the differential equation

$$\frac{d\zeta}{dx}=\frac{R(x,t)}{P(x,t)}. \quad \ldots\ldots (38)$$

Substituting Eq. 37 and using the initial condition $\zeta=0$ at $x=0$, we obtain

$$\zeta=\int_0^x\frac{R[x,t_1+F(x)]}{P[x,t_1+F(x)]}dx=f(x,t). \quad \ldots\ldots (39)$$

Eqs. 37 and 38 define a one-parameter family of characteristics in the x,t,ζ space with the parameter t_1. By varying t_1, we generate a surface $\zeta(x,t)$ that represents the required solution. The equation of this surface is obtained by eliminating t_1 between Eqs. 37 and 39. We obtain

$$\zeta(x,t)=f[x,t-F(x)] \quad \ldots\ldots (40)$$

as the required solution for Eq. 23.

We have assumed that the fluid temperature at the crack entrance is constant. However, this method still applies when the entrance temperature is time-dependent, as would be the case for variable injection rate. In this case, we denote

$$-\theta_0(0,t)=\theta_1(t) \quad \ldots\ldots (41)$$

to be the variable difference between formation temperature and fluid temperature at $x=0$. We take

$$\zeta_1(t)=\log\left[\frac{\theta_r}{\theta_1(t)}\right]. \quad \ldots\ldots (42)$$

The integration procedure is the same as before except that Eq. 39 is replaced by

$$\zeta=\zeta_1+\int_0^x\frac{R[x,t_1+F(x)]}{P[x,t_1+F(x)]}dx=\zeta_1(t)+f(x,t_1). \quad \ldots\ldots (43)$$

The required solution of Eq. 23 is then

$$\zeta(x,t)=\zeta_1(t)+f[x,t-F(x)]. \quad \ldots\ldots (44)$$

Practical Application

The variational method developed here can be used with any fracture propagation theory to determine fluid-temperature distributions. The variational results can be combined through an iterative procedure with the propagation equations to give self-consistent results for crack dimensions and fluid temperature.

Here we combine the 2D Lagrangian analysis of fracture propagation[7] with the variational method. We consider only the case of constant wellbore temperature. The procedure is as follows. First we assume constant temperature at all x and t and use the Lagrangian method to obtain $w(x,t)$, $v(x,t)$, and $q(x,t)$.

As a starting point, it is convenient to assume a constant fluid temperature about 20°F [11°C] below reservoir temperature. The Lagrangian analysis provides $\alpha(x)$ and $t_0(x)$ results for the fluid

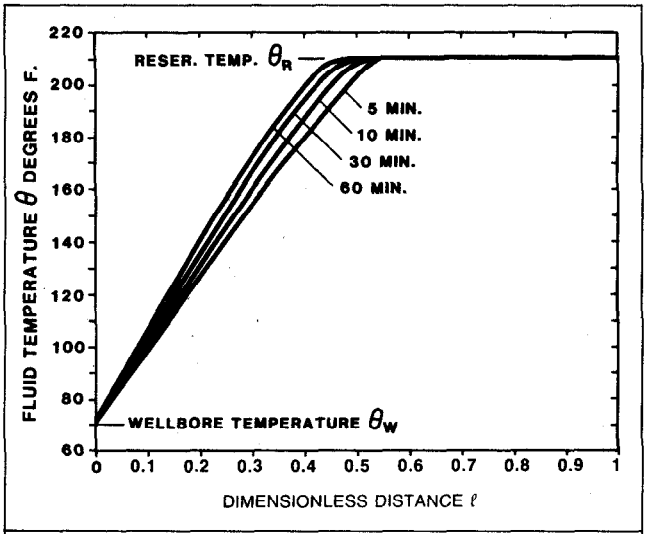

Fig. 4—Temperature profiles given by data of Table 1.

properties corresponding to this temperature.[7] These results are used with Eqs. 26 and 27 to obtain $D_p(x,t)$.

We next find $\zeta(x,t)$ by solving Eq. 23 by the method of characteristics. Eqs. 33 give $P(x,t)$, $T(x,t)$, and $R(x,t)$. We use these results to solve Eq. 34 and then Eq. 38 for each x and t. We then find $\zeta(x,t)$ from Eq. 40.

From $\zeta(x,t)$, we obtain the temperature distribution $\theta_0(x,t)$ from Eqs. 28 and 29. Using this temperature distribution, we repeat the Lagrangian analysis to obtain a new $w(x,t)$, $v(x,t)$, and $q(x,t)$. In this second analysis, we account for the effects of temperature on fluid properties at each x and t. These effects are significant only for non-Newtonian gels where they are expressed through the power-law parameters n and K. This is most conveniently handled by means of a table of values of n and K vs. temperature.

The new results for $w(x,t)$, $v(x,t)$, and $q(x,t)$ are used as before to find a new temperature distribution $\theta_0(x,t)$. This procedure is repeated until convergence is obtained in the results for w, v, q, D_p, and θ_0.

The final results provide a family of temperature profiles like those shown in Fig. 4. The profiles presented here were obtained with the values listed in Table 1 for the important fracturing parameters. The heat capacity, C_s, and thermal conductivity, k_s, of the rock matrix are typical values for sandstones. C_s varies little in different rock materials,[8] but k_s can vary by more than a factor of 10 in typical varieties of reservoir rock.[9] The mixed heat capacity, C_m, and thermal conductivity, k_m, were determined from Eqs. 3 and 5.

TABLE 1—PARAMETERS TO COMPUTE PROFILES OF FIG. 4

Thermal Parameters	
C', cal/(g-°C)	1.0
C_f, cal/(g-°C)	1.0
C_s, cal/(g-°C)	55
k_f, cal/(cm-s-°C)	0.0015
k_s, cal/(cm-s-°C)	0.005
ϕ	0.15
θ_w, °F	70
θ_R, °F	210

Treatment Parameters	
Injection rate, bbl/min-ft of gross fracture height	1
Net/gross interval	0.3
Formation permeability, md	0.01
Far field effective stress, psi	2,000
Fluid spurt loss	0
Fluid wall-building loss, $cm^3/min^{1/2}\cdot cm^2$	0.2
Reservoir fluid viscosity, cp	0.02
Reservoir fluid compressibility, psi^{-1}	5.0×10^{-4}
Reservoir rock Young's modulus, psi	5×10^6
Reservoir rock surface tension, ergs/cm^2	10^5

Fluid Rheology

θ (°F)	n	K (lbf-sec^n/ft^2)
70	0.410	0.525
80	0.412	0.522
90	0.417	0.515
100	0.420	0.495
110	0.429	0.470
120	0.436	0.435
130	0.446	0.400
140	0.453	0.370
150	0.460	0.370
160	0.468	0.365
170	0.470	0.355
180	0.470	0.345
190	0.470	0.335
200	0.470	0.325
210	0.470	0.315

The profiles of Fig. 4 are presented as fluid temperature vs. dimensionless distance, $\ell=x/L$. Plotted in this way, the profiles change very little with time. As time passes, the leading edge of the profile moves very slowly toward smaller ℓ. At a given ℓ, the fracturing fluid is either at reservoir temperature or increasing slowly toward it in time. This is consistent with the mechanics of heat transfer in a growing fracture. Constant ℓ corresponds to a point moving away from the wellbore in time toward the region of reservoir temperature.

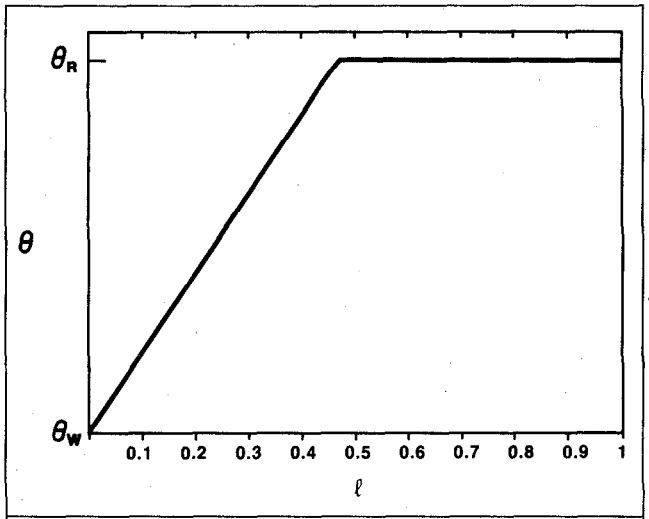

Fig. 5—Approximate profile used to represent the curves of Fig. 4.

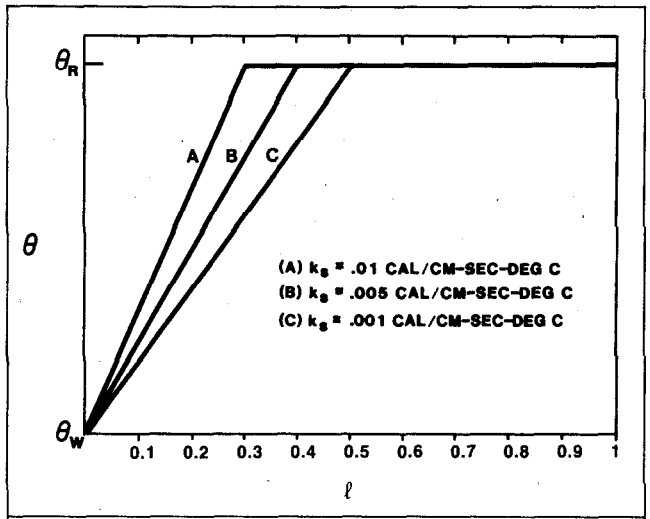

Fig. 6—Approximate profiles representing a large variety of fracturing conditions.

As a practical approximation, the single profile of Fig. 5 can be used to represent all but the very early times in Fig. 4. This approximate profile consists of two straight-line segments. One is drawn from wellbore temperature, θ_w, at $\ell=0$ to reservoir temperature θ_R at $\ell=0.47$. The other is a line drawn along θ_R from $\ell=0.47$ to $\ell=1.0$.

We find that the temperature profile of Fig. 5 is surprisingly general. As long as it is plotted in terms of θ_w, θ_R, and ℓ, it seems to be relatively insensitive to all the parameters in Table 1 except k_s, including n and K characteristics. We find roughly the same profile in going from the most viscous crosslinked gels to very thin ones. Similarly, there is little sensitivity to parameters that control leakoff. We find little difference between profiles for high- and for low-efficiency treatments.

Variations in k_s produce a shift in the value of ℓ at which θ reaches θ_R. This is illustrated in the approximate profiles shown in Fig. 6 for a range of k_s values typical of common reservoir rock.[9] As a first approximation, all can be represented by the profile labeled B in this figure.

For field applications, the B profile of Fig. 6 should be accurate enough to estimate fluid temperature in nearly all fracturing treatments. It predicts a very simple result. The fluid temperature increases linearly with ℓ from wellbore temperature to reservoir temperature, reaching the latter about halfway out in the fracture. This condition prevails throughout the treatment except during very early times.

These approximations apply only to reservoirs with average values of matrix conductivity, k_s. This would include nearly all sandstone and limestone formations. They would not apply to formations with unusually low k_s. One example is the diatomaceous earth formation of southern California. Here, bulk thermal conductivities are on the order of 0.0001 cal/cm-s-°C. Profiles for treatments in this formation reach θ_R at ℓ values between 0.8 and 0.9.

Conclusions

Thermal analysis of hydraulic fracturing can be treated by the variational formulation of convective heat transfer. Results already developed for heat convection and conduction in mixed solid/fluid systems can be applied.

The fracturing problem can be treated in a straightforward manner. First, a simple result can be derived to express the variational principle for heat transfer in a porous solid with fluid convection. This result can be used to develop a relationship between surface heat flow and temperature. These results lead to a differential equation for temperature distribution in the fracture with fracture dimensions and leakoff distribution treated as known functions. Fracture dimensions and leakoff can be determined initially by any 2D analysis that ignores temperature effects. These results, used in the thermal analysis, provide an initial temperature distribution. This distribution can be used to account for fluid viscosity dependence on temperature. The resulting viscosity distribution can be used for a second evaluation of fracture dimensions and leakoff. This iterative procedure can be carried to convergence usually within three or four iterations.

The thermal analysis of a fracture treatment can be considered in terms of a profile of fluid temperature, θ, vs. dimensionless fracture length, ℓ. These profiles can be normalized by relating them to wellbore temperature, θ_w, and reservoir temperature, θ_R. Considered in this way, the profile of a fracture treatment changes little with time over a variety of fracturing conditions and formation properties. In general, θ rises almost linearly from θ_w at $\ell=0$ to θ_R at $\ell=½$ and remains constant at θ_R to the crack tip $\ell=1$.

This practical approximation is very useful for field operations. It provides a rapid and fairly accurate means of estimating fracture temperatures during a treatment.

Nomenclature

b = numerical constant introduced in Eq. C-10 and evaluated in Eqs. C-15 and C-19 to have the average value 0.5472

c = proportionality constant introduced in Eqs. C-11 and C-16

C' = volumetric heat capacity of fluid in the fracture

C_f = heat capacity of pore-filling fluid

C_m = mixed volumetric heat capacity of fluid-filled rock material

C_s = volumetric heat capacity of solid rock matrix

C_1 = constant of integration introduced in Eq. 26

D = dissipation function introduced in Eq. A-7 by putting Eq. 8 in Lagrangian form

D_p = penetration depth for heat flow

$\dot{D}_p$ = derivative of D_p with respect to time or rate of change of penetration depth

f = function introduced to express integral of Eq. 38

F = function introduced to express integral of Eq. 36

H = heat content per unit area of porous rock in x-z plane

$\dot{H}$ = rate of heat flow by convection and conduction per unit area of porous rock material in y direction

H_0 = value of H at the surface $y=0$

$\dot{H}_0$ = value of $\dot{H}$ at the surface $y=0$

J = conductive part of Q_h given by Eq. 2

k_f = thermal conductivity of the fracturing fluid

k_m = mixed thermal conductivity of fluid-filled rock material

k_s = thermal conductivity of the solid rock matrix

K = consistency index in power-law relation for non-Newtonian fluids

ℓ = dimensionless distance along fracture, x/L

L = fracture length from wellbore to tip

n = flow-behavior index in power-law relation for non-Newtonian fluids

P = function defined by Eq. 33

q = volumetric fluid flow rate in the fracture

Q = function defined by Eq. 9

R = function defined by Eq. 33

t = time variable

t_0 = time at which fluid front has reached Location x in fracture

t_1 = constant of integration introduced in Eq. 37

T = function defined by Eq. 33

v = volumetric velocity of fluid injected by leakoff into unit area of fracture face

V = function defined by Eq. 10

w = width distribution along fracture

x,y,z = Cartesian coordinates oriented with x along direction of fracture propagation and y perpendicular to fracture face

α = leakoff function defined by Eq. 25

ζ = variable defined by Eq. D-5

$\dot{\zeta}$ = time derivative of ζ

ζ_1 = variable defined by Eq. 42

θ = temperature of fracturing fluid

$\dot{\theta}$ = time derivative of θ

θ_a = arbitrarily chosen positive temperature

θ_f = excess formation temperature above the fluid temperature at the fracture entrance

θ_r = arbitrarily chosen positive reference temperature introduced in Eq. D-5

θ_R = reservoir temperature

θ_w = constant wellbore temperature

θ_0 = temperature of fracturing fluid at fracture face surface, $y=0$

$\dot{\theta}_0$ = time derivative of θ_0

θ_1 = variable difference between the formation temperature and the fluid temperature at the fracture entrance, $x=0$

ϕ = porosity

Acknowledgment

We thank Mobil R&D Corp. for permission to publish this paper.

References

1. Whitsitt, N.F. and Dysart, G.R.: "The Effect of Temperature on Stimulation Design," *JPT* (April 1970) 493–502, *Trans.*, AIME, **249**.
2. Wheeler, J.A.: "Analytical Calculations of Heat Transfer from Fractures," paper SPE 2494 presented at the 1962 Improved Oil Recovery Symposium, Tulsa, OK, April 12–13.
3. Sinclair, A.R.: "Heat Transfer Effects in Deep Well Fracturing," *JPT* (Dec. 1971) 1484–92; *Trans.*, AIME, **251**.
4. Poulsen, D.K. and Lee, W.S.: "Fracture Design with Time and Temperature Dependent Fluid Properties," paper SPE 12483 presented at the 1984 Formation Damage Control Symposium, Bakersfield, CA, Feb. 13–14.
5. Biot, M.A.: "Variational Principles in Heat Transfer," Oxford Press, New York City (1970).
6. Ramey, H.J. Jr.: "Wellbore Heat Transmission," *JPT* (April 1962) 427–35; *Trans.*, AIME, **225**.
7. Biot, M.A., Massé, L., and Medlin, W.L.: "A Two-Dimensional Theory of Fracture Propagation," *SPEPE* (Jan. 1986) 17–30.
8. *Handbook of Physical Constants*, F. Birch, J.F. Schairer, and H.C. Spicer (eds.), GSA special papers, No. 36 (1942).
9. Robertson, E.C.: "Thermal Conductivity of Rocks," USGS Report 79-356 (1979).
10. Carslaw, H.S. and Jaeger, J.C.: "Conduction of Heat in Solids," Oxford Press, New York City (1959) 32, 59, 75.

Appendix A—Expression of Variational Principle

We substitute the value of J from Eq. 1 into the first term of Eq. 7 and integrate the second term by parts. Making use of Eq. 6, we obtain

$$\int_0^{D_p}\left[\frac{1}{k_m}(\dot{H}-vC_f\theta)\delta H+C_m\theta\delta\theta\right]dy=\theta_0\delta H_0. \quad \text{(A-1)}$$

We assume an approximate heat distribution

$$H=H(y,D_p). \quad \text{(A-2)}$$

Treating D_p as an unknown function of time, we write

$$\delta H=\frac{\partial H}{\partial D_p}\delta D_p, \quad \text{(A-3a)}$$

$$\delta H_0=\frac{\partial H_0}{\partial D_p}\delta D_p, \quad \text{(A-3b)}$$

$$\theta=-\frac{1}{C_m}\frac{\partial H}{\partial D_p}, \quad \text{(A-3c)}$$

and

$$\delta\theta=\frac{\partial\theta}{\partial D_p}\delta D_p. \quad \text{(A-3d)}$$

Substituting these equations into Eq. A-1 gives Eq. 8.

As a matter of interest, Eq. 8 can be put into a standard Lagrangian form.[5] Because

$$\dot{H}=\frac{\partial H}{\partial D_p}\dot{D}_p, \quad \text{(A-4)}$$

we can write

$$\frac{\partial\dot{H}}{\partial\dot{D}_p}=\frac{\partial H}{\partial D_p}. \quad \text{(A-5)}$$

Substituting this result in Eq. 8 gives

$$\frac{\partial D}{\partial\dot{D}_p}+\frac{\partial V}{\partial D_p}=Q, \quad \text{(A-6)}$$

where D is the dissipation function,

$$D=\frac{1}{2k_m}\int_0^{D_p}(\dot{H}-vC_f\theta)^2dy. \quad \text{(A-7)}$$

This formulation can be extended to any number, N, of generalized coordinates by assuming H to be a function of all of them. This gives N equations of the form of Eq. A-6.

Appendix B—Derivation of Terms in Eq. 8

Substituting Eq. 11 into the conservation equation, Eq. 6, gives

$$H=\int_y^{D_p}C_m\theta dy=\tfrac{1}{2}C_m\theta_0D_p\left(1-\frac{y}{D_p}\right)^2. \quad \text{(B-1)}$$

At $y=0$,

$$H_0=\tfrac{1}{2}C_m\theta_0D_p. \quad \text{(B-2)}$$

The time derivative is

$$\dot{H}=\tfrac{1}{2}\dot{D}_pC_m\theta_0\left(1-\frac{y^2}{D_p^2}\right)+\tfrac{1}{2}D_pC_m\dot{\theta}_0\left(1-\frac{y}{D_p}\right)^2. \quad \text{(B-3)}$$

At $y=0$, this reduces to Eq. 12.

From Eq. B-1, we find

$$\frac{\partial H}{\partial D_p}=\tfrac{1}{2}C_m\theta_0\left(1-\frac{y^2}{D_p^2}\right). \quad \text{(B-4)}$$

We use these results to evaluate the integral term in Eq. 8:

$$\frac{1}{k_m}\int_0^{D_p}(\dot{H}-vC_f\theta)\frac{\partial H}{\partial D_p}dy=\frac{2C_m^2\theta_0^2}{15k_m}D_p\dot{D}_p$$

$$-\frac{5v}{24}\frac{C_mC_f}{k_m}\theta_0^2D_p+\frac{3C_m^2}{40k_m}\theta_0\dot{\theta}_0D_p^2. \quad \text{(B-5)}$$

Substituting Eq. 11 into Eq. 10 and differentiating gives

$$\frac{\partial V}{\partial D_p}=\tfrac{1}{6}C_m\theta_0^2. \quad \text{(B-6)}$$

Substituting Eq. B-2 into Eq. 9 gives the needed expression for Q:

$$Q=\tfrac{1}{2}C_m\theta_0. \quad \text{(B-7)}$$

We substitute Eqs. B-5 through B-7 into Eq. 8 to get Eq. 13.

Appendix C—Accuracy Evaluations

We wish to evaluate the accuracy of Eq. 13 as an expression of the variational principle. We consider the case of a constant temperature, θ_0, applied suddenly at the surface $y=0$ at time $t=0$. Thus $\dot{\theta}_0=0$. We also assume there is no convective heat flow, i.e., $v=0$. Eq. 13 then reduces to

$$2D_p\dot{D}_p=5\frac{k_m}{C_m}. \quad \text{(C-1)}$$

Hence,

$$D_p=\sqrt{\frac{5k_mt}{C_m}}. \quad \text{(C-2)}$$

Substituting Eq. C-2 into Eq. 12, we find the rate of heat flow into the solid at its surface to be

$$\dot{H}_0=\frac{\sqrt{5}}{4}\theta_0\sqrt{\frac{k_mC_m}{t}}=0.5590\theta_0\sqrt{\frac{k_mC_m}{t}}. \quad \text{(C-3)}$$

The exact value for this case is given by Carslaw and Jaeger[10] as

$$\dot{H}_0=\frac{1}{\sqrt{\pi}}\theta_0\sqrt{\frac{k_mC_m}{t}}=0.5641\theta_0\sqrt{\frac{k_mC_m}{t}}. \quad \text{(C-4)}$$

The agreement in this case is within 1%.

The next case of interest is $\theta_0=$constant and $v\neq 0$. In this case, Eq. 13 gives

$$2D_p\dot{D}_p-\frac{25vC_fD_p}{8C_m}=5\frac{k_m}{C_m}. \quad \text{(C-5)}$$

For large t, we get large D_p, and the solution tends asymptotically to

$$2\dot{D}_p=\frac{25vC_f}{8C_m}. \quad \text{(C-6)}$$

By integration,

$$D_p=\frac{25vC_ft}{16C_m}. \quad \text{(C-7)}$$

Using Eq. 12 again, we find the rate of heat flow to be

$$\dot{H}_0=^{25}\!/_{32}vC_f\theta_0. \quad \text{(C-8)}$$

Because heat flow in this case is entirely convective, we know the exact result to be

$$\dot{H}_0=vC_f\theta_0. \quad \text{(C-9)}$$

Eq. C-8 would give the exact result if we replaced v by $32v/25$. Making this change in Eq. C-5 gives the revised expression cited as Eq. 14.

Eq. 14 is still a good approximation for the case where θ is time-dependent. Adopting it for this case is equivalent to neglecting the term $9\dot{\theta}_0D_p^2/8\theta_0$ in Eq. 13. Dropping this term can be justified in several ways. First, we note that $\dot{\theta}_0/\theta_0$ is the logarithmic derivative and hence tends to be small. Also, Eq. 13 was derived by a method valid essentially when the absolute magnitude of θ_0 monotonically increases with time. Here we are interested in the opposite condition. Furthermore, Eq. 13 yields D_p, which is little dependent on the temperature variation, θ_0, at the boundary. On the basis of these arguments, we adopt the simplified and corrected Eq. 14 as the basic one for D_p.

We can use a similar procedure to improve Eq. 12. First, we rewrite this equation as

$$\dot{H}_0=\tfrac{1}{2}C_m\dot{D}_p\theta_0+\tfrac{1}{2}bC_mD_p\dot{\theta}_0. \quad \text{(C-10)}$$

Here, b is a numerical coefficient to be determined by best matching $\dot{H}_0$ to exact values for cases where θ_0 is time-dependent and $v=0$.

Consider the case

$$\theta_0=c\sqrt{t}, \quad \text{(C-11)}$$

with $v=0$. Eq. 14 gives

$$D_p=\sqrt{\frac{5k_mt}{C_m}}. \quad \text{(C-12)}$$

And from Eq. C-10, we get

$$\dot{H}_0=\frac{\sqrt{5}}{4}(1+b)c\sqrt{k_mC_m}. \quad \text{(C-13)}$$

In this case, the exact value is known to be[10]

$$\dot{H}_0=\frac{\sqrt{\pi}}{2}c\sqrt{k_mC_m}. \quad \text{(C-14)}$$

Eqs. C-13 and C-14 are equivalent if

$$b=0.5852. \quad \text{(C-15)}$$

We now consider the case

$$\theta_0=ct, \quad v=0. \quad \text{(C-16)}$$

For D_p, we again obtain Eq. C-12. Eq. C-10 gives

$$\dot{H}_0=\frac{\sqrt{5}}{2}(b+\tfrac{1}{2})c\sqrt{k_mC_mt}. \quad \text{(C-17)}$$

The exact result for this case is obtained from Eq. C-4 by use of Duhamel's integral.[10] By this means, we obtain

$$\dot{H}_0=\frac{2c}{\sqrt{\pi}}\sqrt{k_mC_mt}. \quad \text{(C-18)}$$

Eqs. C-17 and C-18 are equivalent for

$$b=0.5092. \quad \text{(C-19)}$$

As a practical approximation, we use the average of Eqs. C-15 and C-19 in Eq. C-10 to obtain Eq. 15.

Appendix D—Conservation-of-Energy Equation

Expansion of Eq. 22 gives

$$C'q\frac{\partial\theta_0}{\partial x}+C'\theta_0\frac{\partial q}{\partial x}+2\dot{H}_0+C'w\frac{\partial\theta_0}{\partial t}+C'\theta_0\frac{\partial w}{\partial t}=0. \quad \text{(D-1)}$$

Substituting for $\partial w/\partial t$ from Eq. 20 gives

$$C'q\frac{\partial\theta_0}{\partial x}+2\dot{H}_0+C'w\frac{\partial\theta_0}{\partial t}-2vC'\theta_0=0. \quad \text{(D-2)}$$

Substituting for $\dot{H}_0$ from Eq. 15, we get

$$C'q\frac{\partial\theta_0}{\partial x}+C_m\theta_0\frac{\partial D_p}{\partial t}+(0.5472C_mD_p+C'w)\frac{\partial\theta_0}{\partial t}-2vC'\theta_0=0. \quad \text{(D-3)}$$

This result can be simplified to

$$-\frac{C'q}{\theta_0}\frac{\partial\theta_0}{\partial x}-C_m\frac{\partial D_p}{\partial t}-(0.5472C_mD_p+C'w)\left(\frac{1}{\theta_0}\right)\frac{\partial\theta_0}{\partial t}$$
$$+2vC'=0. \quad \text{(D-4)}$$

In practical cases, the fluid temperature is always less than the formation temperature. Because θ_0 is the excess temperature above formation temperature, its values are negative, and we can write

$$\zeta(x,t)=\log\left(-\frac{\theta_a}{\theta_0}\right), \quad \text{(D-5)}$$

where θ_a is an arbitrarily chosen positive temperature. In terms of the variable ζ, Eq. D-4 becomes

$$C'q\frac{\partial\zeta}{\partial x}+(0.5472C_mD_p+C'w)\frac{\partial\zeta}{\partial t}-C_m\frac{\partial D_p}{\partial t}+2vC'=0. \quad \text{(D-6)}$$

Eq. D-6 is linear in ζ. We can find D_p by integrating Eq. 14 with the initial condition $D_p=0$ at $t=t_0$. By eliminating $\partial D_p/\partial t$ between Eqs. D-6 and 14, we obtain Eq. 23.

SI Metric Conversion Factors

cp	× 1.0*	E−03	=	Pa·s
erg/cm^2	× 1.0*	E+00	=	mJ/m^2
ft^2	× 9.290 304*	E−02	=	m^2
°F	(°F−32)/1.8		=	°C
kcal/(kg-°C)	× 4.184*	E+00	=	J/g·K
lbf	× 4.448 222	E+00	=	N
psi	× 6.894 757	E+00	=	kPa
psi^{-1}	× 1.450 377	E−01	=	kPa^{-1}

*Conversion factor is exact.

JPT

Original SPE manuscript received for review Sept. 16, 1984. Paper accepted for publication Feb. 27, 1987. Revised manuscript received Aug. 14, 1987. Paper (SPE 13228) first presented at the 1984 SPE Annual Technical Conference and Exhibition held in Houston, Sept. 1984.